Fractal Analysis of Hyperbolic Saddles with Applications

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Minkowski dimension

Definition

Let $S \subset \mathbb{R}^N$ *be a bounded set. For a positive real number* δ*, let* $S_\delta := \{x \in \mathbb{R}^N : d(x, S) < \delta\}$*, and let* $|S_\delta|$ *denote the Lebesgue measure of the set* S_δ*. If the limit*

$$\lim_{\delta \to 0} N - \frac{\log |S_\delta|}{\log \delta}$$

exists, we say that the Minkowski dimension of S*,* $\dim_B S$*, is equal to it.*

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In general, it is not easy to explicitly express $|S_\delta|$.

Example (Nucleus-Tail method)

Minkowski dimension of convergent sequences

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Lemma

Let α be a positive real number. The Minkowski dimension of the sequence $\left(\frac{1}{n^\alpha}\right)_n$ is $\frac{1}{1+\alpha}$.

Minkowski dimension of spiral trajectories

Theorem (Žubrinić, Županović)

Let's consider the normal form of the Hopf-Takens bifurcation

$$\begin{cases} \dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}) \\ \dot{\varphi} = 1 \end{cases} .$$

Let Γ *be a part of a trajectory of this system near the origin.*

- (a) *Assume that* $a_0 \neq 0$. *Then the spiral* Γ *is of exponential type, that is, comparable with* $r = e^{a_0 \varphi}$ *and hence* $dim_B \Gamma = 1$
- (b) *Let* k *be fixed,* $1 \leq k \leq l$, $a_l = 1$ *and* $a_0 = ... = a_{k-1} = 0$, $a_k \neq 0$. *Then* Γ *is comparable with the spiral* $r = \varphi^{-\frac{1}{2k}}$, *and*

$$dim_B \Gamma = \frac{4k}{2k+1}$$

Minkowski dimension of sprial trajectories

Theorem (Žubrinić, Županović)

If the system from the previous theorem has a limit cycle $r = a$ of multiplicity $m, 1 \leq m \leq l$. By Γ_1 and Γ_2 we denote the parts of two trajectories of this system near the limit cycle from the outside and inside resprectively. Then the trajectories Γ_1 and Γ_2 are comparable

1. *with exponential spirals $r = a \pm e^{-\beta\varphi}$ of limit cycle type when $m = 1$, for some constant $\beta > 0$*
2. *with power spirals $r = a \pm \varphi^{-\frac{1}{m-1}}$ when $m > 1$.*

In both cases we have

$$dim_B \Gamma_i = 2 - \frac{1}{m}, \ \ i = 1, 2$$

Theorem (Elezović, Žubrinić, Županović)

Let $\alpha > 1$ and let $f : (0, r) \to (0, +\infty)$ be a monotonically nondecreasing function such that $f(x) \simeq x^{\alpha}$ as $x \to 0$, and $f(x) < x$ for all $x \in (0, r)$. Consider the sequence $S(x_0) := (x_n)_{n \in \mathbb{N}_0}$ defined by

$$x_{n+1} = x_n - f(x_n),\ x_0 \in (0, r).$$

Then

$$x_n \simeq n^{-\frac{1}{\alpha - 1}} \text{ as } n \to \infty.$$

Furthermore,

$$dim_B S(x_0) = 1 - \frac{1}{\alpha},$$

and the set $S(x_0)$ is Minkowski nondegenerate.

Fractal analysis of polycycles

- Fractal analysis near singularities

Fractal analysis of polycycles

- Fractal analysis near singularities
- Fractal analysis near regular sides of the polycycle (away from the singularities)

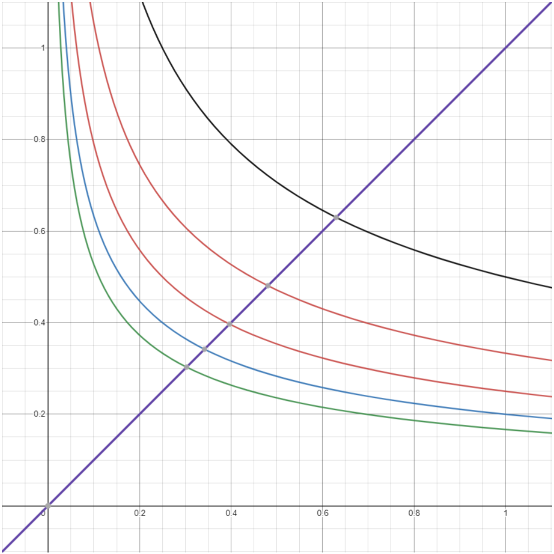
Example

Let's consider the vector field

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \alpha y \end{cases}, \alpha \in (0, 1).$$

Let $(y_n)_{n\in\mathbb{N}}$ *be a stricly decreasing sequence of positive real numbers that converges to* 0 *and let* $(\Gamma_n)_{n\in\mathbb{N}}$ *be the integral curves of the system above that pass through points* $(1, y_n)$. *We define the sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(z_n)_{n\in\mathbb{N}}$ *so that points* $(x_n, 1)$ *and* (z_n, z_n) *lie on* Γ_n.
What is the connection between the Minkowski dimension of the three sequences?

Fractal analysis of a hyperbolic saddle



Example

Let $y_n = \frac{1}{n}$. *Then* $x_n = \frac{1}{n^{\frac{1}{\alpha}}}$ *and* $z_n = \frac{1}{n^{\frac{1}{1+\alpha}}}$ *and*

$$dim_B(x_n)_n < dim_B(y_n)_n < dim_B(z_n)_n.$$

Lemma (C., Huzak, Resman)

For $\alpha \in (0,1]$ *we consider the vector field*

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \alpha y \end{cases} .$$

Let $(y_n)_n$ *be a sequence of positive real numbers that converges monotonically to* 0 *and such that the difference between consecutive members decrease. If the sequence* $(y_n)_n$ *had a defined Minkowski dimension* d*, then the family* $(\Gamma_n)_n$ *of parts of trajectories of the vector field starting at points* $(1, y_n)$ *and ending on* $\{y = 1\}$ *has Minkowski dimension* $1 + d$.

Fractal analysis of the saddle loop

First return map to any transversal to the saddle loop has the form

$$P(x) \simeq x^r,\, x \in \mathbb{R} \setminus \{1\}$$

in codimension 1 case and the form $P(x) = x + \delta(x)$ where

$$\delta(x) = \beta_1 x + \alpha_2 x^2(-\ln x) + \beta_2 x^2 + \ldots + \beta_{k-1} x^{k-1} + \alpha_k x^k(-\ln x) + O(x^k)$$

in higher codimension cases.

Theorem (C., Huzak, Resman)

Consider an analytic vector field X_0 having a saddle-loop. The Minkowski dimension of spiral trajectories that have the loop as their α/ω-limit set depends only on the codimension of the saddle-loop. More precisely, if $k \geq 1$ is the codimension of the saddle loop then the Minkowski dimension of any fixed spiral trajectory S accumulating on it is

$$dim_B S = \begin{cases} 2 - \frac{2}{k}, & k \text{ even}, \\ 2 - \frac{2}{k+1}, & k \text{ odd}. \end{cases}$$

Theorem (C., Huzak, Resman)

Let $X = X_0$ be an analytic vector field with a hyperbolic 2-cycle Γ_2 with irrational ratios of hyperbolicity r_1 and r_2 such that $r_1 r_2 = 1$. Then the upper bound on the cyclicity of Γ_2 under perturbations of X can be read from the Minkowski dimension of spiral trajectories of the vector field X accumulating on Γ_2. More precisely, if the dimension of a spiral trajectory is $d < 2$, then the cyclicity of Γ_2 under perturbations of X is not greater than

$$3 + (1 + r)\frac{d - 1}{2 - d}$$

where $1 > r \in \{r_1, r_2\}$.

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Remark

Spiral trajectories accumulating on "simpler" 2-cycles always have Minkowski dimension equal to 1.

Thank you!