Iterated Fractal Drums ~ Some New Perspectives:

Polyhedral Measures, Atomic Decompositions

Joint work with Michel L. Lapidus

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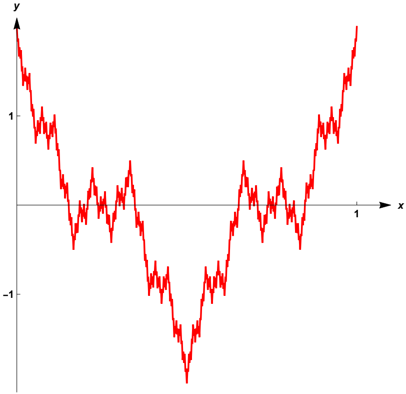
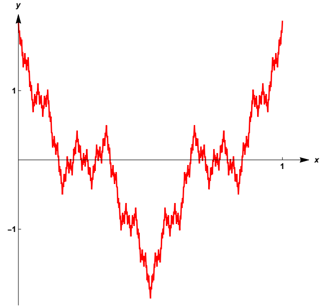
Introduction

A pathological object





$$x \in \mathbb{R} \mapsto \mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos(\pi b^n x) \quad , \quad 0 < \lambda < 1 \quad , \quad \lambda b > 1$$

Continuous everywhere, while being nowhere differentiable[I],[II].

[I] Karl Weierstrass. "Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differential quotienten besitzen". In: *Journal für die reine und angewandte Mathematik* 79 (1875), pp. 29–31.

[II] Godfrey Harold Hardy. "Weierstrass's Non-Differentiable Function". In: *Transactions of the American Mathematical Society* 17.3 (1916), pp. 301–325.

Minkowski Dimension[III], [IV], [V], [VI]:

$$D_{\mathscr{W}} = 2 + \frac{\ln \lambda}{\ln b} = 2 - \ln_b \frac{1}{\lambda}$$

[III] James L. Kaplan, John Mallet-Paret, and James A. Yorke. "The Lyapunov dimension of a nowhere differentiable attracting torus". In: *Ergodic Theory and Dynamical Systems* 4 (1984), pp. 261–281.

[IV] Feliks Przytycki and Mariusz Urbański. "On the Hausdorff dimension of some fractal sets". In: *Studia Mathematica* 93.2 (1989), pp. 155–186.

[V] Tian-You Hu and Ka-Sing Lau. "Fractal Dimensions and Singularities of the Weierstrass Type Functions". In: *Transactions of the American Mathematical Society* 335.2 (1993), pp. 649–665.

[VI] Claire David. "Bypassing dynamical systems: A simple way to get the box-counting dimension of the graph of the Weierstrass function". In: *Proceedings of the International Geometry Center* 11.2 (2018), pp. 1–16. URL: https://journals.onaft.edu.ua/index.php/geometry/article/view/1028.

Our question:

Can we find

A suitable measure?

I. The Geometric Framework

We hereafter place ourselves in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x, y). The horizontal and vertical axes will be respectively refered to as $(x'x)$ and $(y'y)$.

Notation

In the following, λ and N_b are two real numbers such that:

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 \cdot$$

We consider the *Weierstrass function* $\mathscr{W}$, defined, for any real number x, by

$$\mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \cdot$$

Associated graph: **the Weierstrass Curve**.

Due to **the one-periodicity** of the $\mathscr{W}$ function, we restrict our study to the interval $[0, 1[$.

Minkowski (or box-counting) Dimension

$$D_{\mathscr{W}} = 2 + \frac{\ln \lambda}{\ln N_b},$$ *equal to its Hausdorff dimension* [VII, VIII, IX, X]

[VII] James L. Kaplan, John Mallet-Paret, and James A. Yorke. "The Lyapunov dimension of a nowhere differentiable attracting torus". In: *Ergodic Theory and Dynamical Systems* 4 (1984), pp. 261–281.

[VIII] Krzysztof Barański, Balázs Bárány, and Julia Romanowska. "On the dimension of the graph of the classical Weierstrass function". In: *Advances in Mathematics* 265 (2014), pp. 791–800.

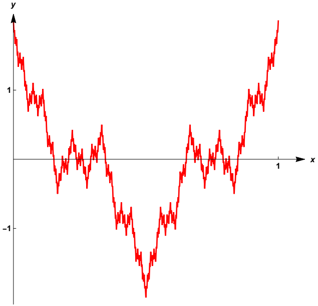
[IX] Weixiao Shen. "Hausdorff dimension of the graphs of the classical Weierstrass functions". In: *Mathematische Zeitschrift* 289 (1-2 2018), pp. 223–266.

[X] Gerhard Keller. "A simpler proof for the dimension of the graph of the classical Weierstrass function". In: *Annales de l'Institut Henri Poincaré – Probabilités et Statistiques* 53.1 (2017), pp. 169–181.

The Weierstrass Curve as a Cyclic Curve

In the sequel, we identify the points

$$(0, \mathscr{W}(0)) \quad \text{and} \quad (1, \mathscr{W}(1)) = (1, \mathscr{W}(0)) \cdot$$



Remark

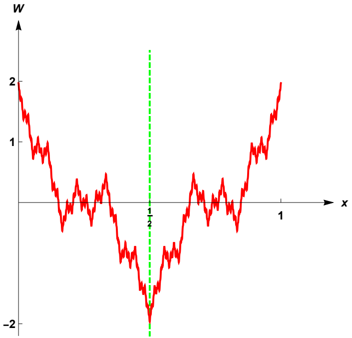
The above convention makes sense, in so far as the points $(\mathbf{0}, \mathscr{W}(\mathbf{0}))$ and $(\mathbf{1}, \mathscr{W}(\mathbf{1}))$ have **the same vertical coordinate**, in addition to the periodic properties of the $\mathscr{W}$ function.

Property (Symmetry with respect to the vertical line $x = \frac{1}{2}$)

Since, for any $x \in [0,1]$*:*

$$\mathcal{W}(1-x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\pi N_b^n - 2\pi N_b^n x\right) = \mathcal{W}(x)$$

the Weierstrass Curve is **symmetric with respect to the vertical straight line** $x = \frac{1}{2}$.



Proposition (Nonlinear and Noncontractive Iterated Function System (IFS))

We approximate the restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the Weierstrass Curve, by a sequence of finite graphs, built through an iterative process, by using **the nonlinear iterated function system** *(IFS) of the family of C^∞ maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ denoted by*

$$\mathcal{T}_{\mathcal{W}} = \left\{ T_0, \cdots, T_{N_b-1} \right\} ,$$

where, for $0 \leq i \leq N_b - 1$ and any point (x,y) of $\mathbb{R}^2$,

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\,\pi \left(\frac{x+i}{N_b} \right) \right) \right) \cdot$$

Property (Attractor of the IFS)

The Weierstrass Curve is **the attractor** *of the IFS $\mathcal{T}_{\mathcal{W}}$:* $\boldsymbol{\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})}$.

Fixed Points

For any integer i belonging to $\{0, \cdots, N_b - 1\}$, we denote by:

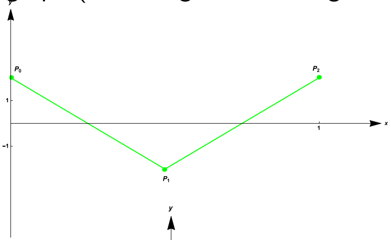
$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\,\pi\, i}{N_b - 1} \right) \right)$$

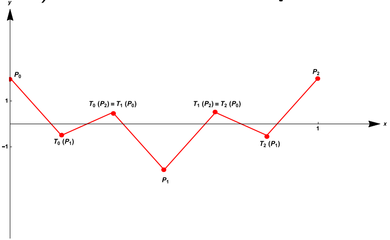
the fixed point of the map $\boldsymbol{T_i}$.

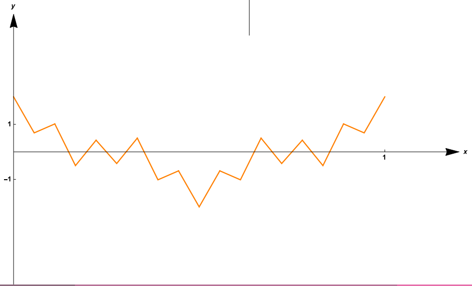
Sets of vertices, Prefractals

We set: $\boldsymbol{V_0} = \{\boldsymbol{P_0}, \cdots, \boldsymbol{P_{N_b-1}}\}$, and, for any $m \in \mathbb{N}^\star$: $\boldsymbol{V_m} = \bigcup\limits_{i=0}^{N_b-1} \boldsymbol{T_i}\,(\boldsymbol{V_{m-1}})$.

For $m \in \mathbb{N}$, the set of points V_m, where two consecutive points are linked, is an oriented graph (according to increasing abscissa): the $\boldsymbol{m^{th}}$**-order** $\mathscr{W}$**-prefractal** $\Gamma_{\mathscr{W}_m}$.

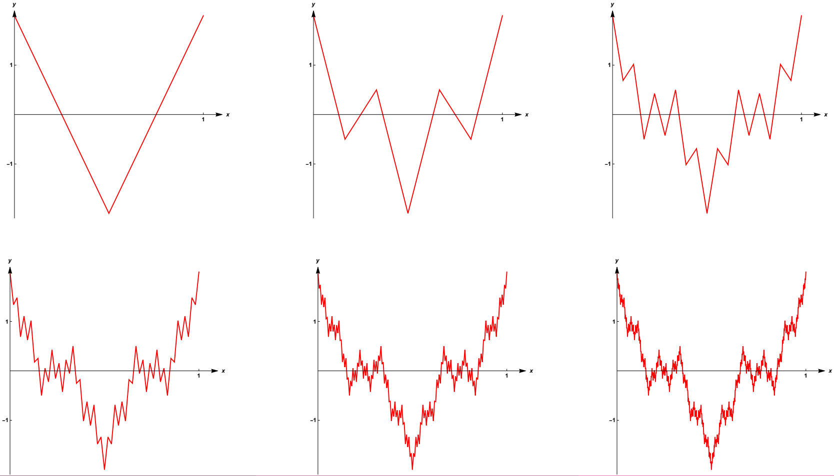






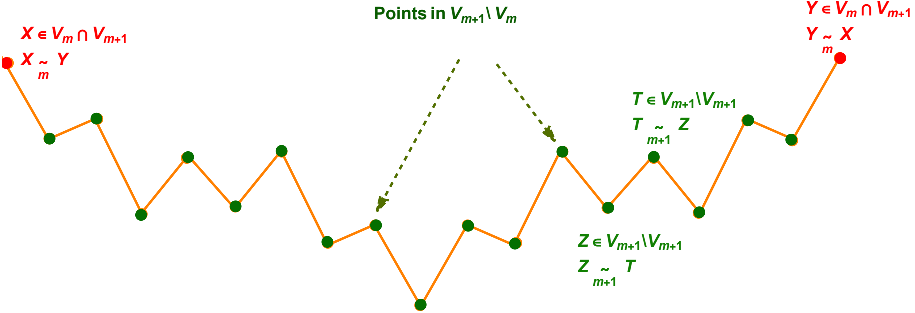
The Weierstrass IFD

We call **Weierstrass Iterated Fractal Drums (IFD)** the sequence of prefractal graphs which converge to the Weierstrass Curve.



Adjacent Vertices, Edge Relation

For any natural integer m, the prefractal graph $\Gamma_{\mathscr{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$: two vertices X and Y of $\Gamma_{\mathscr{W}_m}$, i.e. two points belonging to V_m, will be said to be **adjacent** (i.e., neighboring or junction points) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathscr{W}_m}$; we then write $X \underset{m}{\sim} Y$. **This edge relation depends on m**, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.

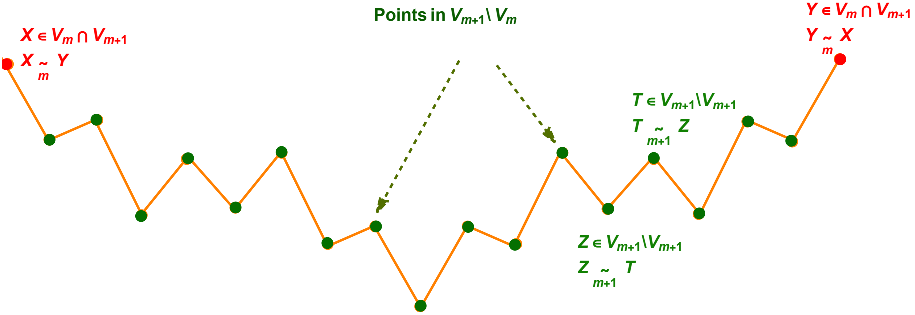


Property

For any natural integer m, we have that

i. $V_m \subset V_{m+1}$.

ii. $\# V_m = (N_b - 1)\, N_b^m + 1$.

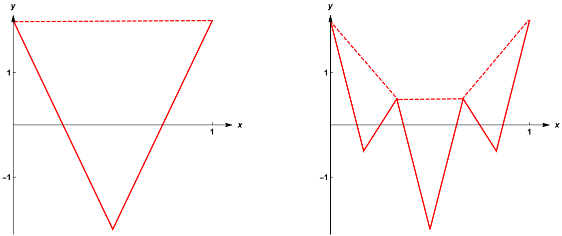


iii. The prefractal graph $\Gamma_{\mathscr{W}_m}$ has exactly $(N_b - 1)\, N_b^m$ edges.

iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathscr{W}_m}$ are the vertices of N_b^m **simple polygons** $\mathscr{P}_{m,k}$ with N_b sides. For $m \in \mathbb{N}$, the junction point between two consecutive polygons is the point

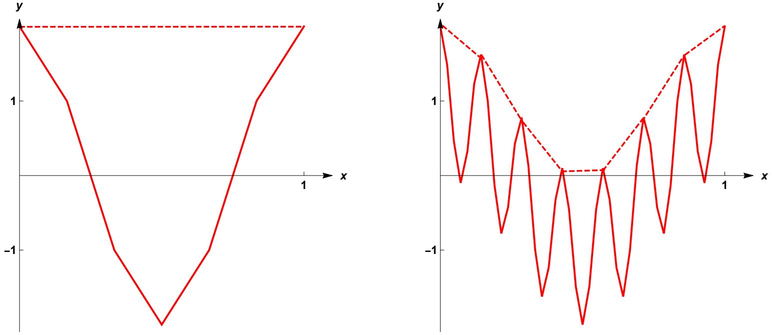
$$\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathscr{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m} \right) \right) \quad , \quad 1 \leq k \leq N_b^m - 1 \cdot$$

The total number of junction points is thus $N_b^m - 1$.
For instance, in the case $N_b = 3$, one gets triangles.
In the sequel, we will denote by $\mathscr{P}_{\mathbf{0}}$ **the initial polygon**, i.e. the one whose vertices are the fixed points of the maps T_i, $0 \leq i \leq N_b - 1$.



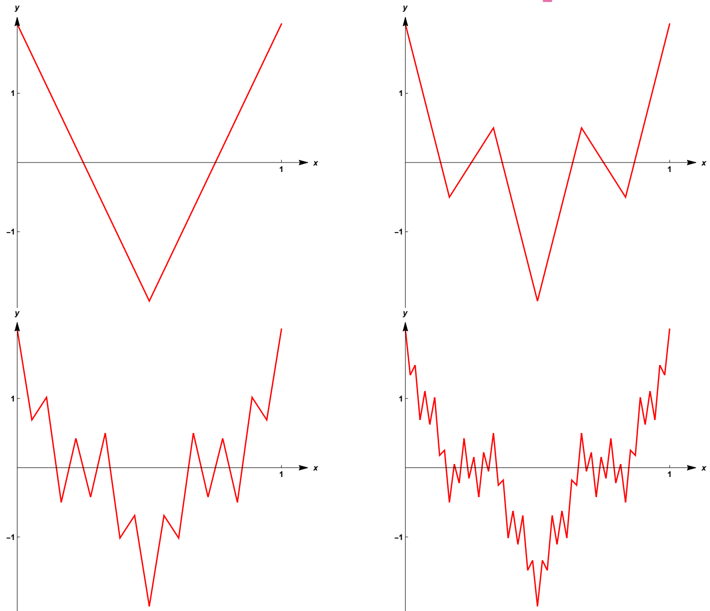
The polygons, in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.

The polygons, in the case where $\lambda = \frac{1}{2}$, and $N_b = 7$.

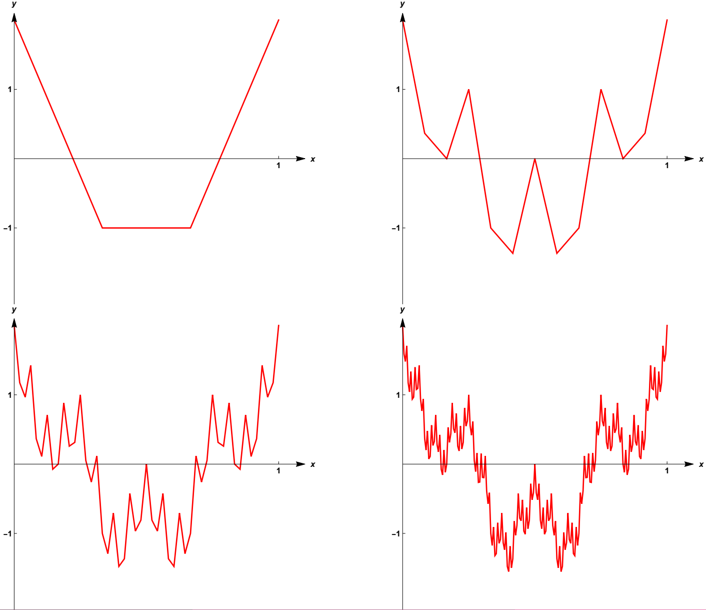


$m = 0$ $m = 1$

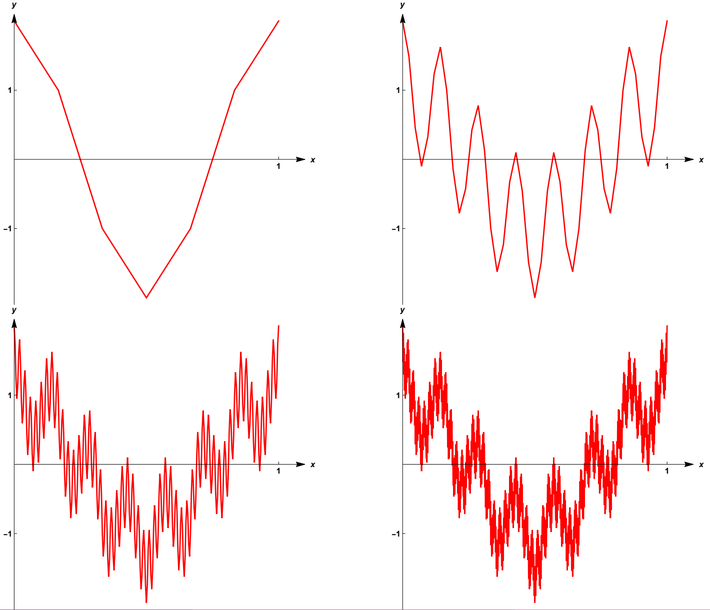
The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.



The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 4$.



The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 7$.



Vertices of the Prefractals, Elementary Lengths, and Heights

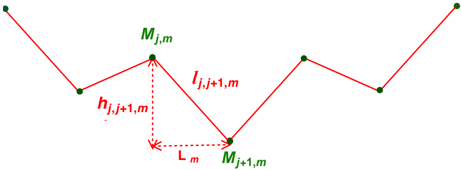
Given $m \in \mathbb{N}$, we denote by $\boldsymbol{\left(M_{j,m}\right)_{0\leq j\leq (N_b-1)\,N_b^m-1}}$ **the set of vertices** of the prefractal graph $\Gamma_{\mathscr{W}_m}$. One thus has, for any integer j in $\left\{0,\cdots,(N_b-1)\,N_b^m-1\right\}$:

$$M_{j,m}=\left(\frac{j}{(N_b-1)\,N_b^m},\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right).$$

We also introduce, for $0\leq j\leq (N_b-1)\,N_b^m-2$:

i. <u>the elementary horizontal lengths</u>:

$$L_m=\frac{1}{(N_b-1)\,N_b^m}$$

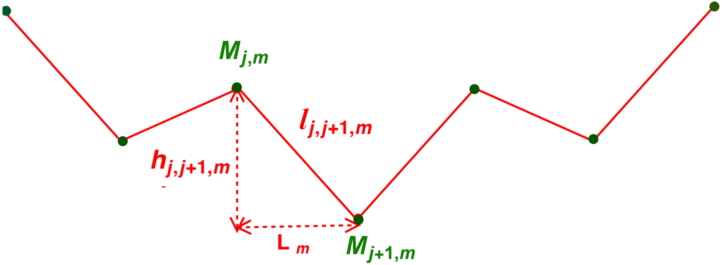


ii. the elementary lengths:

$$\ell_{j,j+1,m} = d\left(M_{j,m}, M_{j+1,m}\right) = \sqrt{L_m^2 + h_{j,j+1,m}^2}$$

iii. the elementary heights:

$$h_{j,j+1,m} = \left|\mathscr{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right) - \mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right|$$



iv. <u>the geometric angles</u>:

$$\theta_{j-1,j,m} = \left((y'y), (\widehat{M_{j-1,m} M_{j,m}})\right) \quad , \quad \theta_{j,j+1,m} = \left((y'y), (\widehat{M_{j,m} M_{j+1,m}})\right),$$

which yield **the value of the geometric angle between consecutive edges** $\left[M_{j-1,m}\, M_{j,m}, M_{j,m}\, M_{j+1,m}\right]$:

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{\left|h_{j-1,j,m}\right|} + \arctan \frac{L_m}{\left|h_{j,j+1,m}\right|} \cdot$$

Property (Scaling Properties of the Weierstrass Function, and Consequences)

Since, for any real number x

$$\mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\pi N_b^n x\right)$$

one also has

$$\mathscr{W}(N_b x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\pi N_b^{n+1} x\right) = \frac{1}{\lambda} \sum_{n=1}^{+\infty} \lambda^n \cos\left(2\pi N_b^n x\right) = \frac{1}{\lambda} \left\{\mathscr{W}(x) - \cos(2\pi x)\right\}$$

which yield, for any strictly positive integer m, and any j in $\{0, \cdots, \# V_m\}$*:*

$$\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda\,\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}\right) + \cos\left(\frac{2\pi j}{(N_b-1)\,N_b^{m-1}}\right)$$

By induction, one obtains that

$$\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)=\lambda^m\,\mathscr{W}\left(\frac{j}{(N_b-1)}\right)+\sum_{k=0}^{m-1}\lambda^k\,\cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right).$$

A Consequence of the Symmetry with respect to the Vertical Line $x = \frac{1}{2}$

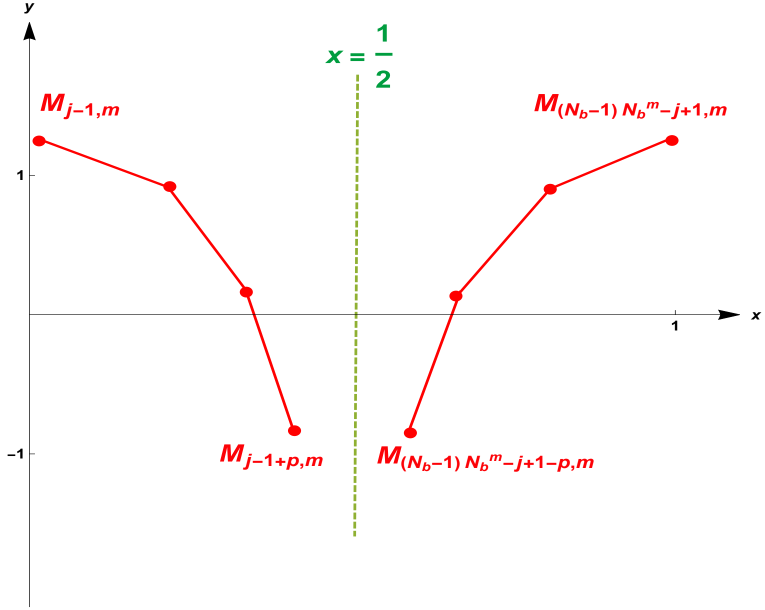
For any strictly positive integer m and any j in $\{0, \cdots, \# V_m\}$, we have that

$$\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \mathscr{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right)$$

which means that **the points**

$$\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}, \mathscr{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right)\right) \quad \text{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m}, \mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x = \frac{1}{2}$.

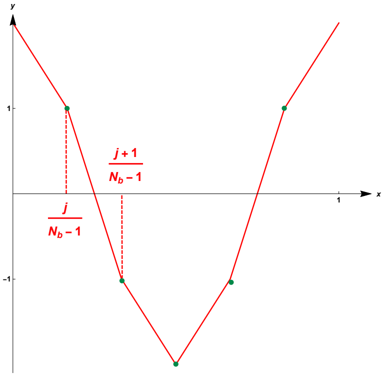


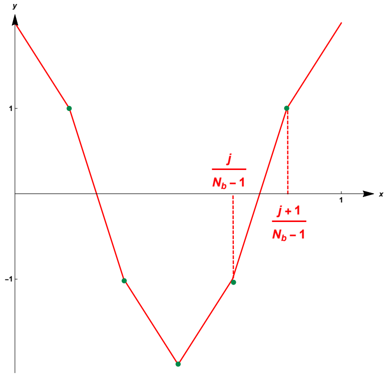
y
$x = \frac{1}{2}$
$M_{j-1,m}$
$M_{(N_b-1)\,N_b{}^m-j+1,m}$
1
1
x
−1
$M_{j-1+p,m}$
$M_{(N_b-1)\,N_b{}^m-j+1-p,m}$

Property

i. **For** $0 \leq j \leq \frac{(N_b - 1)}{2}$**:** $\mathscr{W}\left(\frac{j+1}{N_b-1}\right) - \mathscr{W}\left(\frac{j}{N_b-1}\right) \leq 0.$

ii. **For** $\frac{(N_b - 1)}{2} \leq j \leq N_b - 1$**:** $\mathscr{W}\left(\frac{j+1}{N_b-1}\right) - \mathscr{W}\left(\frac{j}{N_b-1}\right) \geq 0.$





Property

Given a strictly positive integer m:

i. For any j in $\{0, \cdots, \# V_m\}$, the point

$$\left(\frac{j}{(N_b-1)\, N_b^m}, \mathscr{W}\left(\frac{j}{(N_b-1)\, N_b^m} \right) \right)$$

is the image of the point

$$\left(\frac{j}{(N_b-1)\, N_b^{m-1}} - i, \mathscr{W}\left(\frac{j}{(N_b-1)\, N_b^{m-1}} - i \right) \right) = \left(\frac{j-i\,(N_b-1)\, N_b^{m-1}}{(N_b-1)\, N_b^{m-1}}, \mathscr{W}\left(\frac{j-i\,(N_b-1)\, N_b^{m-1}}{(N_b-1)\, N_b^{m-1}} \right) \right)$$

by the map T_i, $0 \leq i \leq N_b - 1$.

As a consequence, **the j^{th} vertex of the polygon** $\mathscr{P}_{m,k}$, $0 \leq k \leq N_b^m - 1$, $0 \leq j \leq N_b - 1$, i.e. the point:

$$\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m}, \mathscr{W}\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m} \right) \right)$$

is the image of the point

$$\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}}, \mathscr{W}\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}} \right) \right)$$

i.e. is the **the j^{th} vertex of the polygon** $\mathscr{P}_{m-1,k-i\,(N_b-1)\,N_b^{m-1}}$.
There is thus **an exact correspondence between vertices of the polygons at consecutive steps $m-1$, m**.

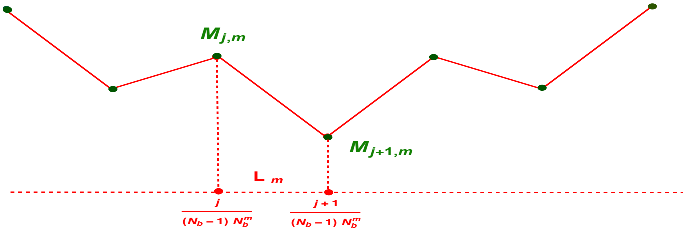
ii. Given j in $\{0,\cdots,N_b-2\}$, and k in $\left\{0,\cdots,N_b^m-1\right\}$:

$$\text{sign}\left(\mathscr{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m} \right) - \mathscr{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m} \right) \right) = \text{sign}\left(\mathscr{W}\left(\frac{j+1}{N_b-1} \right) - \mathscr{W}\left(\frac{j}{N_b-1} \right) \right).$$

Bounding Result: Upper and Lower Bounds for the Elementary Heights

For any strictly positive integer m, and any j in $\left\{0, \cdots, (N_b - 1)\, N_b^m\right\}$, we have that

$$C_{inf} \underbrace{\lambda^m}_{N_b^{m\,(D_{\mathcal{W}}-2)}} \leq \left| \mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) \right| \leq C_{sup} \underbrace{\lambda^m}_{N_b^{m\,(D_{\mathcal{W}}-2)}}$$



where

$$C_{inf} = (N_b - 1)^{2-D_{\mathscr{W}}} \min_{0 \leq j \leq N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathscr{W}}} \left(\max_{0 \leq j \leq N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right| + \frac{2\,\pi}{(N_b - 1)\,(\lambda\, N_b - 1)} \right).$$

These constants depend on the initial polygon $\mathscr{P}_0$.

Theorem: Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function

For any natural integer m, and any pair of real numbers (x, x') such that:

$$x = \frac{(N_b - 1)\,k + j}{(N_b - 1)\,N_b^m} = ((N_b - 1)\,k + j)\;L_m \quad , \quad x' = \frac{(N_b - 1)\,k + j + \ell}{(N_b - 1)\,N_b^m} = ((N_b - 1)\,k + j + \ell)\;L_m$$

where $0 \leq k \leq N_b - 1^m - 1$, and

i. if the integer N_b is odd,

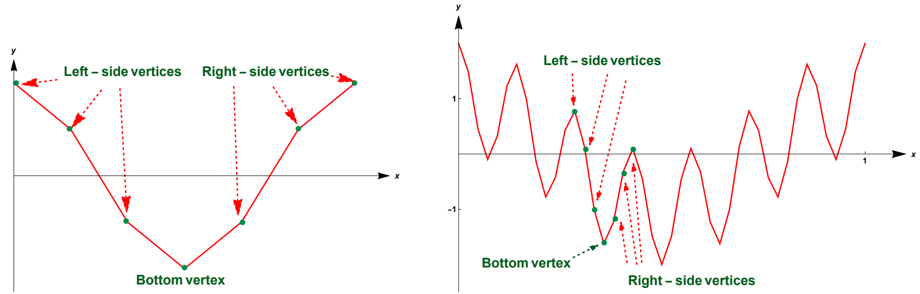
$$0 \leq j < \frac{N_b - 1}{2} \quad \text{and} \quad 0 < j + \ell \leq \frac{N_b - 1}{2}$$

$$\text{or} \quad \frac{N_b - 1}{2} \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b - 1}{2} < j + \ell \leq N_b - 1\,;$$

ii. if the integer N_b is even,

$$0 \leq j < \frac{N_b}{2} \quad \text{and} \quad 0 < j + \ell \leq \frac{N_b}{2}$$

$$\text{or} \quad \frac{N_b}{2} + 1 \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b}{2} + 1 < j + \ell \leq N_b - 1\,,$$



This means that the points $\left(x, \mathscr{W}(x)\right)$ and $\left(x', \mathscr{W}(x')\right)$ are vertices of the polygon $\mathscr{P}_{m,k}$ **both located on the left-side of the polygon**, or **on the right-side**. Then, one has the following *reverse-Hölder inequality*, with **sharp Hölder exponent** $-\dfrac{\ln \lambda}{\ln N_b} = 2 - D_{\mathscr{W}}$,

$$C_{inf} \left| x' - x \right|^{2-D_{\mathscr{W}}} \leq \left| \mathscr{W}(x') - \mathscr{W}(x) \right| \cdot$$

Corollary

One may now write, for any $m \in \mathbb{N}^\star$, and $0 \leq j \leq (N_b - 1)\, N_b^m - 1$:

i. **for the elementary heights**:

$$h_{j-1,j,m} = L_m^{2-D_{\mathscr{W}}}\, \mathscr{O}(1)$$

ii. **for the elementary quotients**:

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathscr{W}}}\, \mathscr{O}(1)$$

where:

$$0 < C_{inf} \leq \mathscr{O}(1) \leq C_{sup} < \infty \cdot$$

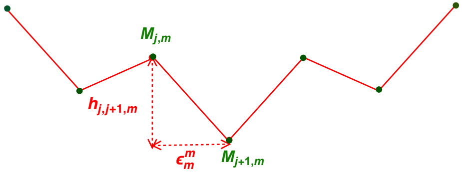
II. Polyhedral Measure

m^{th} Cohomology Infinitesimal

Given any $m \in \mathbb{N}$, we will call *m^{th} cohomology infinitesimal* the number

$$\varepsilon_m^m = \frac{1}{N_b - 1} \, \frac{1}{N_b^m} \underset{m \to \infty}{\to} 0 \cdot$$

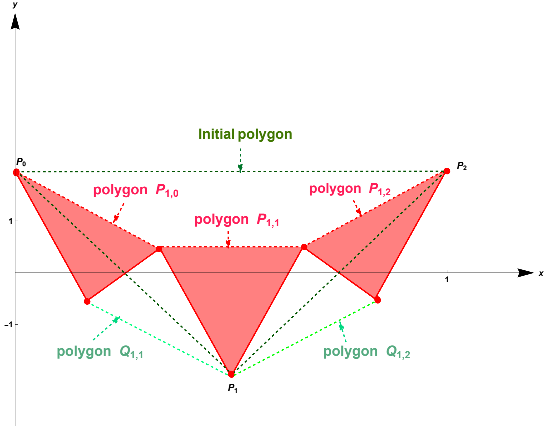
Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of $\mathscr{W}$.



Polygonal Sets

For any $m \in \mathbb{N}$, the consecutive vertices of the prefractal graph $\Gamma_{\mathscr{W}_m}$ are the vertices of N_b^m **simple polygons** $\mathscr{P}_{m,k}$ with N_b sides.
We now introduce the **polygonal sets**

$$\mathscr{P}_m = \left\{ \mathscr{P}_{m,k} \, , \, 0 \leq k \leq N_b^m - 1 \right\} \quad \text{and} \quad \mathscr{Q}_m = \left\{ \mathscr{Q}_{m,k} \, , \, 0 \leq k \leq N_b^m - 2 \right\} \cdot$$



Notation

For any $m \in \mathbb{N}$, we denote by:

ii. $X \in \mathscr{P}_m$ (resp., $X \in \mathscr{Q}_m$) a vertex of a polygon $\mathscr{P}_{m,k}$, with $0 \leq k \leq N_b^m - 1$ (resp., a vertex of a polygon $\mathscr{Q}_{m,k}$, with $1 \leq k \leq N_b^m - 2$).

ii. $\mathscr{P}_m \bigcup \mathscr{Q}_m$ the reunion of the polygonal sets $\mathscr{P}_m$ and $\mathscr{Q}_m$, which consists in the set of all the vertices of the polygons $\mathscr{P}_{m,k}$, with $0 \leq k \leq N_b^m - 1$, along with the vertices of the polygons $\mathscr{Q}_{m,k}$, with $1 \leq k \leq N_b^m - 2$. In particular, $X \in \mathscr{P}_m \bigcup \mathscr{Q}_m$ simply denotes a vertex in $\mathscr{P}_m$ or $\mathscr{Q}_m$.

iii. $\mathscr{P}_m \bigcap \mathscr{Q}_m$ the intersection of the polygonal sets $\mathscr{P}_m$ and $\mathscr{Q}_m$, which consists in the set of all the vertices of both a polygon $\mathscr{P}_{m,k}$, with $0 \leq k \leq N_b^m - 1$, and a polygon $\mathscr{Q}_{m,k'}$, with $1 \leq' k \leq N_b^m - 2$.

Power of a Vertex

Given $m \in \mathbb{N}^\star$, a vertex $\boldsymbol{X}$ of $\boldsymbol{\Gamma}_{\mathscr{W}_m}$ is said:

i. *of* **power one** *relative to the polygonal family* $\mathscr{P}_m$ if X **belongs to** (or is a vertex of) **one and only one** N_b**-gon** $\mathscr{P}_{m,j}$, for $0 \leq j \leq N_b^m - 1$;

ii. *of* **power** $\frac{1}{2}$ *relative to the polygonal family* $\mathscr{P}_m$ if X is **a common vertex to two consecutive** N_b**-gons** $\mathscr{P}_{m,j}$ and $\mathscr{P}_{m,j+1}$, for $0 \leq j \leq N_b^m - 2$;

iii. *of* **power zero** *reative to the polygonal family* $\mathscr{P}_m$ if X **does not belong to** (or is not a vertex of) **any** N_b**-gon** $\mathscr{P}_{m,j}$, for $0 \leq j \leq N_b^m - 1$.

Similarly, given $m \in \mathbb{N}$, a vertex $\boldsymbol{X}$ of $\boldsymbol{\Gamma}_{\mathscr{W}_m}$ is said:

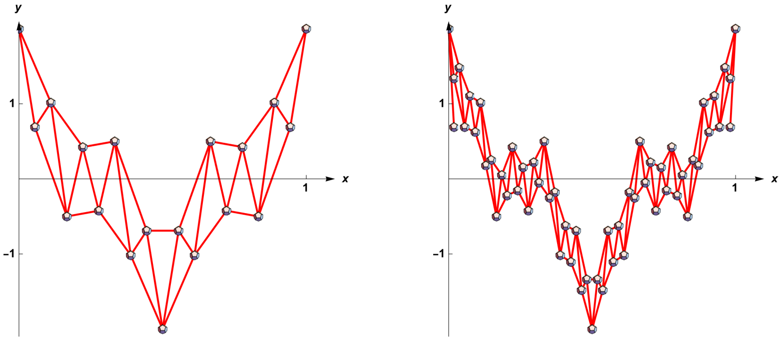
i. *of* **power one** *relative to the polygonal family* $\mathscr{Q}_m$ if X **belongs to** (or is a vertex of) **one and only one** N_b**-gon** $\mathscr{P}_{m,j}$, for $0 \leq j \leq N_b^m - 2$;

ii. *of* **power** $\frac{1}{2}$ *relative to the polygonal family* $\mathscr{P}_m$ if X is **a common vertex to two consecutive** N_b**-gons** $\mathscr{Q}_{m,j}$ and $\mathscr{Q}_{m,j+1}$, for $0 \leq j \leq N_b^m - 3$;

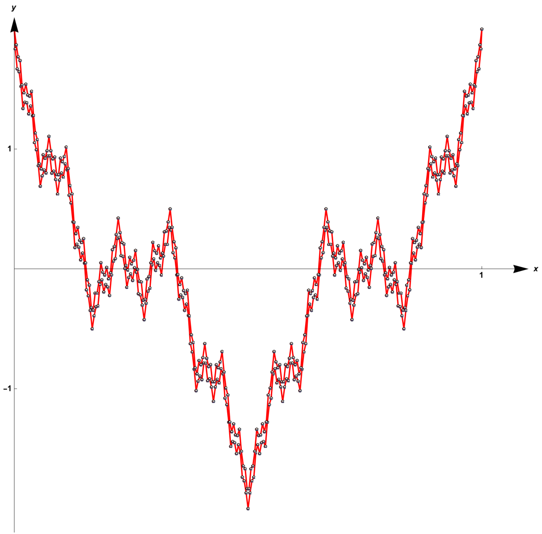
iii. *of* **power zero** *reative to the polygonal family* $\mathscr{P}_m$ if X **does not belong to** (or is not a vertex of) **any** N_b**-gon** $\mathscr{Q}_{m,j}$, for $0 \leq j \leq N_b^m - 2$.

Sequence of Domains Delimited by the $\mathscr{W}$ IFD

We introduce *the sequence of domains delimited by the Weierstrass IFD* as the sequence $\left(\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $\left(\mathscr{P}_m \cup \mathscr{Q}_m\right)_{m\in\mathbb{N}}$, where, for each $m \in \mathbb{N}$, $\mathscr{P}_m$ and $\mathscr{Q}_m$ respectively denote the polygonal sets introduced just above.



$\mathscr{D}\left(\Gamma_{\mathscr{W}_2}\right)$ **and** $\mathscr{D}\left(\Gamma_{\mathscr{W}_3}\right)$**, for** $\lambda = \dfrac{1}{2}$ **and** $N_b = 3$**.**



$\mathscr{D}\left(\Gamma_{\mathscr{W}_5}\right)$, for $\lambda = \dfrac{1}{2}$ and $N_b = 3$.

Domain Delimited by the Weierstrass IFD

We call *domain, delimited by the Weierstrass IFD*, the set, which is equal to the following limit,

$$\mathscr{D}\left(\Gamma_{\mathscr{W}}\right) = \lim_{m\to\infty} \mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right) ,$$

where the convergence is interpreted **in the sense of the Hausdorff metric** on $\mathbb{R}^2$. In fact, we have that

$$\mathscr{D}\left(\Gamma_{\mathscr{W}}\right) = \Gamma_{\mathscr{W}} \cdot$$

Notation (Lebesgue Measure (on $\mathbb{R}^2$))

In the sequel, we denote by $\mu_{\mathscr{L}}$ the Lebesgue measure on $\mathbb{R}^2$.

Notation

For any $m \in \mathbb{N}$, and any vertex X of V_m, we set:

$$\mu^{\mathscr{L}}(X, \mathscr{P}_m, \mathscr{Q}_m) = \begin{cases} \dfrac{1}{N_b}\, p(X, \mathscr{P}_m) \displaystyle\sum_{0 \leq j \leq N_b^m - 1, X \text{ vertex of } \mathscr{P}_{m,j}} \mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right), \text{ if } X \notin \mathscr{Q}_m, \\ \dfrac{1}{N_b}\, p(X, \mathscr{Q}_m) \displaystyle\sum_{1 \leq j \leq N_b^m - 2, X \text{ vertex of } \mathscr{P}_{m,j}} \mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right), \text{ if } X \notin \mathscr{P}_m, \\ \dfrac{1}{2\,N_b} \left\{ p(X, \mathscr{P}_m) \displaystyle\sum_{\substack{0 \leq j \leq N_b^m - 1, \\ X \text{ vertex of } \mathscr{P}_{m,j}}} \mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right) + p(X, \mathscr{Q}_m) \displaystyle\sum_{\substack{1 \leq j \leq N_b^m - 2, \\ X \text{ vertex of } \mathscr{Q}_{m,j}}} \mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right) \right\} \\ \text{if } X \in \mathscr{P}_m \cap \mathscr{Q}_m \,. \end{cases}$$

Property

We set

$$m_{\mathscr{W}} = \min_{t\in[0,1]} \mathscr{W}(t) \quad , \quad M_{\mathscr{W}} = \max_{t\in[0,1]} \mathscr{W}(t) \cdot$$

Given a continuous function u on $[0,1]\times[m_{\mathscr{W}}, M_{\mathscr{W}}]$, we have that, for any $m \in \mathbb{N}$, and any vertex X of V_m:

$$\left|\mu^{\mathscr{L}}(X,\mathscr{P}_m,\mathscr{Q}_m)\, u(X)\right| \leq \mu^{\mathscr{L}}(X,\mathscr{P}_m,\mathscr{Q}_m) \left(\max_{[0,1]\times[m_{\mathscr{W}}, M_{\mathscr{W}}]} |u|\right) \lesssim N_b^{-(3-D_{\mathscr{W}})\,m} \,.$$

Consequently, we have that

$$\varepsilon_m^{m(D_{\mathscr{W}}-2)} \left|\mu^{\mathscr{L}}(X,\mathscr{P}_m,\mathscr{Q}_m)\, u(X)\right| \lesssim \varepsilon_m^{-m} \cdot$$

Since the sequence $\left(\sum_{X\in\mathscr{P}_m\bigcup\mathscr{Q}_m} \varepsilon_m^{-m}\right)_{m\in\mathbb{N}}$ is **a positive and increasing sequence** (the number of vertices involved increases as m increases), this ensures the existence of the finite limit

$$\lim_{m\to\infty} \varepsilon_m^{m(D_{\mathscr{W}}-2)} \sum_{X\in\mathscr{P}_m\bigcup\mathscr{Q}_m} \mu^{\mathscr{L}}(X,\mathscr{P}_m,\mathscr{Q}_m)\, u(X) \cdot$$

Proof

For any $m \in \mathbb{N}$, and any vertex X of V_m, we have that

$$\mu^{\mathscr{L}}(X, \mathscr{P}_m, \mathscr{Q}_m) \lesssim \varepsilon_m^{m(D_{\mathscr{W}}-3)} \quad \text{and} \quad \mu^{\mathscr{L}}(X, \mathscr{P}_m, \mathscr{Q}_m) \lesssim \varepsilon_m^{m(D_{\mathscr{W}}-3)} .$$

The total number of polygons $\mathscr{P}_m$ is N_b^m, while the total number of polygons $\mathscr{Q}_m$ is equal to $N_b^m - 1$. We then have that

$$\sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}(X, \mathscr{P}_m, \mathscr{Q}_m) \lesssim \varepsilon_m^{m(2-D_{\mathscr{W}})} ,$$

which, as desired, ensures the existence of the finite limit

$$\left(\max_{[0,1]\times[m_{\mathscr{W}}, M_{\mathscr{W}}]} |u| \right) \lim_{m\to\infty} \varepsilon_m^{m(D_{\mathscr{W}}-2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}(X, \mathscr{P}_m, \mathscr{Q}_m) .$$

Polyhedral Measure on the Weierstrass IFD

We introduce **the polyhedral measure** *on the Weierstrass IFD*, denoted by μ, such that for any continuous function u on the Weierstrass Curve,

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\left(X, \mathscr{P}_m, \mathscr{Q}_m\right) u\,(X)\,, \quad (\star)$$

which can also be understood in the following way,

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \int_{\mathscr{D}(\Gamma_{\mathcal{W}})} u\, d\mu\,.$$

Theorem - I

The polyhedral measure μ is **well defined, positive, as well as a bounded, nonzero, Borel measure** on $\mathscr{D}(\Gamma_{\mathscr{W}})$. The associated total mass is given by

$$\mu\left(\mathscr{D}\left(\Gamma_{\mathscr{W}}\right)\right) = \lim_{m\to\infty} \varepsilon_m^{\,m\,(D_{\mathscr{W}}-2)} \sum_{X\,\in\,\mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\left(X, \mathscr{P}_m, \mathscr{Q}_m\right) , \quad (\star\star)$$

and satisfies the following estimate:

$$\mu\left(\mathscr{D}\left(\Gamma_{\mathscr{W}}\right)\right) \leq \frac{2}{N_b}\left(N_b-1\right)^2 C_{sup} \cdot \quad (\star\star\star)$$

Furthermore, the support of μ coincides with the entire curve:

$$\text{supp}\,\mu = \mathscr{D}\left(\Gamma_{\mathscr{W}}\right) = \Gamma_{\mathscr{W}} \cdot$$

Theorem - II

In addition, μ is the weak limit as $m \to \infty$ of *the following* **discrete measures** (or Dirac Combs), given, for each $m \in \mathbb{N}$, by

$$\mu_m = \varepsilon_m^{m\,(D_{\mathscr{W}}-2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}} \left(X, \mathscr{P}_m, \mathscr{Q}_m\right) \delta_X \,,$$

where ε denotes the cohomology infinitesimal, and δ_X the Dirac measure concentrated at X.

Proof ~ i. μ is a well defined measure.

Indeed, the map φ

$$u \mapsto \varphi(u) = \int_{\Gamma_{\mathscr{W}}} u \, d\mu$$

is a well defined linear functional on the space $C(\Gamma_{\mathscr{W}})$ of real-valued, continuous functions on $\Gamma_{\mathscr{W}}$. Hence, by a well-known argument, it is a continuous linear functional on $C(\Gamma_{\mathscr{W}})$, equipped with the *sup* norm. Since $\Gamma_{\mathscr{W}}$ is compact, and in light of its definition, μ is a bounded, Radon measure, with total mass $\varphi(1) = \mu(\mathscr{D}(\Gamma_{\mathscr{W}}))$, also given by $(\star\star)$, and where 1 denotes the constant function equal to 1 on $\Gamma_{\mathscr{W}}$. Then, according to the Riesz representation theorem, the associated positive Borel measure (still denoted by μ) is a bounded and positive Borel measure with the same total mass $\mu(\mathscr{D}(\Gamma_{\mathscr{W}})) = \mu(\Gamma_{\mathscr{W}})$.

Proof ~ *ii*. The nonzero measure – Estimates for the total mass of μ

For $0 \leq j \leq N_b^m - 1$, each polygon $\mathscr{P}_{m,j}$ is contained in a rectangle of height at most equal to $(N_b - 1)\, h_m$, and of width at most equal to $(N_b - 1)\, L_m$. This ensures that the Lebesgue measure of each polygon $\mathscr{P}_{m,j}$ is at most equal to $(N_b - 1)^2\, h_m\, L_m$. We also have the following estimate

$$h_m \leq C_{sup}\, L_m^{2-D_{\mathscr{W}}} \,,$$

where

$$C_{sup} = (N_b - 1)^{2-D_{\mathscr{W}}} \left(\max_{0 \leq j \leq N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right| + \frac{2\,\pi}{(N_b - 1)\,(\lambda\, N_b - 1)} \right) .$$

Consequently:

$$\mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right) \leq (N_b - 1)^2\, C_{sup}\, L_m^{3-D_{\mathscr{W}}} \quad , \quad \mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right) \leq (N_b - 1)^2\, C_{sup}\, L_m^{3-D_{\mathscr{W}}} \,.$$

We then deduce that, for any vertex X of V_m,

$$\mu\,(X,\mathscr{P}_m,\mathscr{Q}_m) \leq \frac{1}{N_b}\,(N_b-1)^2\,C_{sup}\,L_m^{3-D_{\mathscr{W}}}\,.$$

Hence, since the total number of polygons involved is at most equal to $2\,N_b^m-1 \leq 2\,N_b^m$, we can deduce that

$$\sum_{X\in\mathscr{P}_m\bigcup\mathscr{Q}_m} \mu^{\mathscr{L}}\,(X,\mathscr{P}_m,\mathscr{Q}_m) \leq 2\,\frac{\varepsilon_m^{-m}}{N_b}\,(N_b-1)^2\,C_{sup}\,\varepsilon_m^{m\,(3-D_{\mathscr{W}})}\,.$$

We then have that

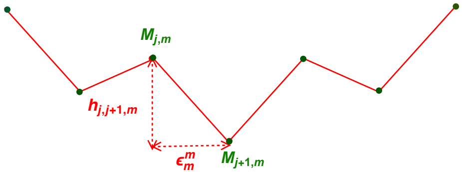
$$\varepsilon_m^{m\,(D_{\mathscr{W}}-2)}\sum_{X\in\mathscr{P}_m\bigcup\mathscr{Q}_m} \mu^{\mathscr{L}}\,(X,\mathscr{P}_m,\mathscr{Q}_m) \leq \frac{2}{N_b}\,(N_b-1)^2\,C_{sup}<\infty\,,$$

from which we can deduce that the polyhedral measure is a bounded measure.

For the sake of simplicity, we restrict ourselves to the case when $N_b < 7$.
For $0 \leq j \leq N_b^m - 1$, each polygon $\mathscr{P}_{m,j}$ (which is convex) contains an inscribed circle, whose Lebesgue measure is greater than $\dfrac{h_m^{inf}\, \varepsilon_m^m}{C_{N_b}}$, where

$$h_m^{inf} = \inf_{0 \leq j \leq (N_b-1)\, N_b^m - 1} h_{j,j+1,m}$$

and where $C_{N_b} > 0$.



We recall that

$$C_{inf}\, \varepsilon_m^{m\,(2-D_{\mathscr{W}})} \leq h_m^{inf} \text{ , where } C_{inf} = (N_b-1)^{2-D_{\mathscr{W}}} \min_{0 \leq j \leq N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b-1}\right) - \mathscr{W}\left(\frac{j}{N_b-1}\right) \right| > 0 \cdot$$

Consequently,

$$\mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right) \geq \frac{h_m^{inf}\, \varepsilon_m^m}{C_{N_b}} \geq \frac{C_{inf}\, \varepsilon_m^{m\,(3-D_{\mathscr{W}})}}{C_{N_b}} \quad , \quad \mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right) \geq \frac{h_m^{inf}\, \varepsilon_m^m}{C_{N_b}} \geq \frac{C_{inf}\, \varepsilon_m^{m\,(3-D_{\mathscr{W}})}}{C_{N_b}} \cdot$$

We then deduce that, for any vertex X of V_m,

$$\mu\,(X, \mathscr{P}_m, \mathscr{Q}_m) \geq \frac{1}{N_b}\,\frac{C_{inf}\,\varepsilon_m^{m\,(3-D_{\mathscr{W}})}}{C_{N_b}}\,.$$

Hence, since the total number of polygons involved is greater than $N_b^m - 1 \geq \frac{N_b^m}{2}$, we can deduce that

$$\sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\,(X, \mathscr{P}_m, \mathscr{Q}_m) \geq \frac{\varepsilon_m^{-m}}{2\,(N_b-1)}\,\frac{C_{inf}\,\varepsilon_m^{m\,(3-D_{\mathscr{W}})}}{N_b\,C_{N_b}}\,.$$

We then have that

$$\varepsilon_m^{m\,(D_{\mathscr{W}}-2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\,(X, \mathscr{P}_m, \mathscr{Q}_m) \geq \frac{1}{2\,(N_b-1)}\,\frac{C_{inf}}{N_b\,C_{N_b}} > 0\,,$$

from which, upon passing to the limit when $m \to \infty$, we can deduce that the polyhedral measure is a nonzero measure, and that its total mass satisfies inequality $(\star\star\star)$.

Proof ∼ *iii*. Supp $\mu = \Gamma_{\mathscr{W}}$

This simply comes from the proof given in *ii*. just above that the measure μ is nonzero. If $u \in C\left(\Gamma_{\mathscr{W}}, \mathbb{R}^+\right)$, we have that

$$\varepsilon_m^{m\,(D_{\mathscr{W}}-2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\,(X, \mathscr{P}_m, \mathscr{Q}_m)\; u(X) \geq \frac{1}{2\,(N_b-1)}\;\frac{C_{inf}}{N_b\,C_{N_b}}\left(\min_{\Gamma_{\mathscr{W}}} u\right) > 0\cdot$$

Hence, upon passing to the limit when $m \to \infty$, we deduce that $\varphi(u) = \displaystyle\int_{\Gamma_{\mathscr{W}}} u\, d\mu > 0$, and thus, $\varphi(u) \neq 0$, from which the claim follows easily.
Indeed, otherwise, if $\operatorname{supp}\mu \neq \Gamma_{\mathscr{W}}$, there exists $M \in \Gamma_{\mathscr{W}} \setminus \operatorname{supp}\mu$, and thus, by Urisohn's lemma (see, e.g.,[XI]), there exists $u \in C\left(\Gamma_{\mathscr{W}}\right)$ and an open neighborhood $\mathscr{V}(M)$ of M in $\Gamma_{\mathscr{W}}$ disjoint from $\operatorname{supp}\mu$ and such that

$$u(M) = 1 \quad , \quad 0 \leq u \leq 1 \quad , \quad \text{and } u_{|\Gamma_{\mathscr{W}} \setminus \mathscr{V}(M)} = 0\cdot$$

Hence, by the above argument, $\varphi(u) \neq 0$, which contradicts the fact that $M \notin \operatorname{supp}\mu$

[XI] Walter Rudin. *Real and Complex Analysis*. Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416. ISBN: 0-07-054234-1.

Proof ~ *iv.* μ is a singular measure

First, note that

$$\mu^{\mathscr{L}}(\Gamma_{\mathscr{W}}) = 0\,,$$

because $D_{\mathscr{W}} < 2$, and, up to a multiplicative positive constant, $\mu^{\mathscr{L}}$ coincides with the 2-dimensional measure on $\mathbb{R}^2$. Now, since $\operatorname{supp} \mu \subset \Gamma_{\mathscr{W}}$, and $\mu^{\mathscr{L}}(\Gamma_{\mathscr{W}}) = 0$, it follows that μ is supported on a set of Lebesgue measure zero, which precisely implies that μ (viewed as a Borel measure on the rectangle $[0,1] \times [m_{\mathscr{W}}, M_{\mathscr{W}}]$ in the obvious way), is singular with respect to the restriction of $\mu^{\mathscr{L}}$ to this rectangle.

Proof - *iv.* μ is the weak limit of the discrete measures μ_m

Indeed, this follows at once from the fact that, for every $u \in \mathscr{C}(\Gamma_{\mathscr{W}})$,

$$\int_{\Gamma_{\mathscr{W}}} u \, d\mu = \lim_{m \to \infty} \int_{\Gamma_{\mathscr{W}}} u \, d\mu_m \,,$$

as desired.

This completes the proof.

The Quasi Self-Similar Sequence of Discrete Polyhedral Measures

The sequence of discrete polyhedral measures $(\mu_m)_{m\in\mathbb{N}}$ introduced just above, satisfies the following recurrence relation, for all $m \in \mathbb{N}^\star$,
The sequence of discrete polyhedral measures $(\mu_m)_{m\in\mathbb{N}}$ introduced in Theorem 53 just above, satisfies the following recurrence relation, for all $m \in \mathbb{N}^\star$,

$$\mu_m = N_b^{D_{\mathscr{W}}-2} \sum_{T_j \in \mathscr{T}_{\mathscr{W}}} \mu_{m+1} \circ T_j^{-1}, \qquad (\spadesuit)$$

where for $\mathscr{T}_{\mathscr{W}} = \left\{T_0, \cdots, T_{N_b-1}\right\}$ is the nonlinear iterated function system (IFS) involved.

Note that relation $(\spadesuit)$ can be viewed as a generalization of classical self-similar measures, as exposed in[XII], page 714.

[XII] John E. Hutchinson. "Fractals and self similarity". In: *Indiana University Mathematics Journal* 30 (1981), pp. 713–747.

Proof

First, we can note that, for $m \in \mathbb{N}^\star$,

$$\varepsilon_{m+1}^{m+1} = \frac{1}{N_b}\, \varepsilon_m^m \,,$$

which ensures that

$$\varepsilon_{m+1}^{(m+1)\,(D_{\mathscr{W}}-2)} = \frac{1}{N_b^{D_{\mathscr{W}}-2}}\, \varepsilon_m^{m\,(D_{\mathscr{W}}-2)} = N_b^{2-D_{\mathscr{W}}}\, \varepsilon_m^{m\,(D_{\mathscr{W}}-2)} \,.$$

We then simply use the result according to which, for $0 \leq j \leq N_b - 1$, the j^{th} vertex of the polygon $\mathscr{P}_{m+1,k}$, $0 \leq k \leq N_b^m - 1$, is the image of the the j^{th} vertex of the polygon $\mathscr{P}_{m,k-i\,(N_b-1)\,N_b^m}$ by the map T_i, where $0 \leq j \leq N_b - 1$ is arbitrary. Therefore, there is an exact correspondance between polygons at consecutive steps m, $m+1$: indeed, polygons at the $(m+1)^{th}$ step of the prefractal approximation process are obtained by applying each map T_i, for $0 \leq i \leq N_b - 1$, to the polygons at the m^{th} step of the prefractal approximation process. We can then deduce that

$$\sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\,(X, \mathscr{P}_m, \mathscr{Q}_{m+})\, \delta_X = \sum_{T_j \in \mathscr{T}_{\mathscr{W}}} \sum_{X \in \mathscr{P}_{m+1} \bigcup \mathscr{Q}_{m+1}} \mu^{\mathscr{L}}\left(X, T_j^{-1}\,(\mathscr{P}_{m+1})\,, T_j^{-1}\,(\mathscr{Q}_{m+1})\right) \delta_X \,,$$

IV. Atomic Decompositions

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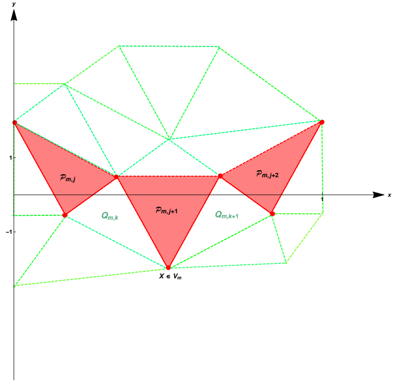
Trace Theorems, and Consequences

Two-Dimensional Polygonal $\pi_{\mathscr{W},m}$-Net, $m \in \mathbb{N}$

Given a strictly positive integer m, we call *two-dimensional polygonal* $\pi_{\mathscr{W},m}$*-net* a tessellation of $\mathbb{R}^2$ into half-open N_b-gons of side lengths at most equal to $\sqrt{2}\, h_m$ which contains the set of polygons

$$\left\{ \bigcup_{j=0}^{N_b^m-1} \mathscr{P}_{m,j} \right\} \bigcup \left\{ \bigcup_{k=1}^{N_b^m-2} \mathscr{Q}_{m,k} \right\} .$$



Property

Given $m \in \mathbb{N}^\star$:

i. For any integer $j \in \left\{0, \cdots, N_b^m - 1\right\}$, and any pair of vertices $(X, Y) \in \left(V_m \cap \mathscr{P}_{m,j}\right)^2$:

$$d_{eucl}(X, Y) \lesssim N_b\, h_m \lesssim N_b^{-m\,(2-D_{\mathscr{W}})}\,.$$

ii. For any integer $j \in \left\{1, \cdots, N_b^m - 2\right\}$, and any pair of vertices $(X, Y) \in \left(V_m \cap \mathscr{Q}_{m,j}\right)^2$:

$$d_{eucl}(X, Y) \lesssim N_b\, h_m \lesssim N_b^{-m\,(2-D_{\mathscr{W}})}\,.$$

Atoms (Generalization of[XIII])

Given $s < 1$, $p > 1$, $m \in \mathbb{N}$ and $j \in \left\{0, \cdots, N_b^m - 1\right\}$, a function $f_{m,j}$ defined on $\Gamma_{\mathscr{W}_m}$ is called a **$(\mathscr{P}_{m,j}, s, p)$-atom** if the following three conditions are satisfied:

i. $\operatorname{Supp} f_{m,j} \subset \mathscr{P}_{m,j}$;

ii. $\forall\, X \in V_m \cap \mathscr{P}_{m,j} : \quad \left|f_{m,j}(X)\right| \lesssim \mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right)^{\frac{s}{D_{\mathscr{W}}} - \frac{1}{p}}$;

iii. $\forall\, (X, Y) \in \left(V_m \cap \mathscr{P}_{m,j}\right)^2$:

$$\left|f_{m,j}(X) - f_{m,j}(Y)\right| \lesssim d_{eucl}(X, Y)\, \mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right)^{\frac{s-1}{D_{\mathscr{W}}} - \frac{1}{p}} .$$

[XIII] M. Kabanava. "Besov Spaces on Nested Fractals by Piecewise Harmonic Functions". In: *Zeitschrift für Analysis und ihre Anwendungen* 31.2 (2012), pp. 183–201.

Similarly, Given $s < 1$, $p > 1$, $m \in \mathbb{N}$ and $j \in \left\{0, \cdots, N_b^m - 1\right\}$, a function $f_{m,j}$ defined on $\Gamma_{\mathscr{W}_m}$ is called a **$(\mathscr{Q}_{m,j}, s, p)$-atom** if the following three conditions are satisfied:

i. $\operatorname{Supp} f_{m,j} \subset \mathscr{Q}_{m,j}\,;$

ii. $\forall\, X \in V_m \cap \mathscr{P}_{m,j}\,: \quad \left|f_{m,j}(X)\right| \lesssim \mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right)^{\frac{s}{D_{\mathscr{W}}}-\frac{1}{p}};$

iii. $\forall\,(X,Y) \in \left(V_m \cap \mathscr{Q}_{m,j}\right)^2\,:$

$$\left|f_{m,j}(X) - f_{m,j}(Y)\right| \lesssim d_{eucl}(X,Y)\,\mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right)^{\frac{s-1}{D_{\mathscr{W}}}-\frac{1}{p}}\,.$$

Atoms Associated with the Weierstrass Function

The restriction of the Weierstrass function to each polygon $\mathscr{P}_{m,j}$, (resp., $\mathscr{Q}_{m,j}$) is a $(\mathscr{P}_{m,j}, s, p)$-*atom* (resp., a $(\mathscr{Q}_{m,j}, s, p)$-*atom*).

Atomic Decomposition of a Function Defined on the Weierstrass Curve

Given a continuous function f on the Weierstrass Curve, we will say that f admits *an atomic decomposition* in the following form:

$$f = \lim_{m\to\infty} \sum_{\mathcal{X} \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \tilde{\lambda}_{f,m,\mathcal{X}}\, \tilde{f}_{m,\mathcal{X}} = \lim_{m\to\infty} \sum_{\mathcal{X} \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \tilde{\lambda}_{f,m}\, \tilde{f}_m\,,$$

where, for any $m \in \mathbb{N}$, we say that $\tilde{\lambda}_{f,m}$ is the m^{th}-*atomic coefficient*.
The functions $\tilde{f}_{m,\mathcal{X}}$ and $\tilde{f}_m$ will be called (m, s, p')-*atoms*.

Atomic Decomposition of Spline Functions

Given $(n, k) \in \mathbb{N}^2$, a spline function of degree k on $\pi_{\mathscr{W},n}$ admits *an atomic decomposition* of the form

$$spline = \lim_{m \to \infty} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \tilde{\lambda}_{s,m,X} \, \widetilde{spline}_{m,X} \quad .$$

(This directly comes from the definition of functions of $\mathscr{P}ol_k \left(\pi_{N_b^n} \right)$ as piecewise polynomial functions.)

Property

Given the polyhedral measure μ on the Weierstrass Curve $\Gamma_{\mathscr{W}}$, and a continuous function f on $\Gamma_{\mathscr{W}}$, of atomic decomposition

$$f = \lim_{m\to\infty} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X} \,,$$

we have that

$$\int_{\mathscr{D}(\Gamma_{\mathscr{W}})} f\, d\mu = \lim_{m\to\infty} \varepsilon^{m\,(D_{\mathscr{W}}-2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X}\, \mu\,(X, \mathscr{P}_m, \mathscr{Q}_m) \cdot$$

Such a decomposition makes sense since the set of vertices $(V_m)_{m\in\mathbb{N}}$ is dense in $\Gamma_{\mathscr{W}}$. Thus, because we deal with continuous functions, given any point X of the Weierstrass Curve, there exists a sequence $(X_m)_{m\in\mathbb{N}}$ such that

$$f(X) = \lim_{m\to\infty} f(X_m)\,,$$

where, for any $m \in \mathbb{N}$, X_m belongs to the prefractal graph $\Gamma_{\mathscr{W}_m}$.
We can naturally write $f(X_m)$ as

$$f(X_m) = \sum_{Y_m \in V_m} f(Y_m)\,\delta_{X_m Y_m}(X_m)\,,$$

where δ is the classical Kronecker symbol; i.e.,

$$\forall\, Y_m \in V_m \,:\quad \delta_{X_m Y_m}(Y_m) = \begin{cases} 1\,, & \text{if} \quad Y_m = X_m\,, \\ 0\,, & \text{else.} \end{cases}$$

This, of course, yields

$$f(X) = \lim_{m\to\infty} \sum_{Y_m \in V_m} f(Y_m)\,\delta_{X_m Y_m}(Y_m)\cdot$$

Now, we can go a little further and, as in[XIV], introduce **spline functions** $\psi^m_{X_m}$ such that

$$\forall\, Y \in \Gamma_{\mathscr{W}} \; : \quad \psi^m_{X_m}(Y) = \begin{cases} \delta_{X_m Y_m}\,, & \forall\, Y \in V_m \\ 0\,, & \forall\, Y \notin V_m\,, \end{cases}$$

and write

$$f(X) = \lim_{m\to\infty} \sum_{Y_m \in V_m} f(Y_m)\, \psi^m_{X_m}(Y_m)\,,$$

which is nothing but the application of **the Weierstrass approximation theorem**. In particular, spline functions are a natural choice for atoms.

[XIV] Robert S. Strichartz. *Differential Equations on Fractals, A tutorial*. Princeton University Press, 2006.

L^p-Norm of a Function on the Weierstrass Curve Defined by Means of an Atomic Decomposition

In the sequel, all functions f considered on the Weierstrass Curve are implicitely supposed to be Lebesgue measurable.

Given $p \in \mathbb{N}^\star$, and a continuous function f on $\Gamma_{\mathcal{W}}$, whose absolute value $|f|$ is defined by means of an atomic decomposition as

$$|f| = \lim_{m \to \infty} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \tilde{\lambda}_{|f|,m,X} \, \widetilde{|f|}_{m,X} \,,$$

its L^p-norm for the measure μ is given by

$$\begin{aligned} \|f\|_{L^p(\Gamma_{\mathcal{W}})} &= \left(\int_{\mathscr{D}(\Gamma_{\mathcal{W}})} |f|^p \, d\mu \right)^{\frac{1}{p}} \\ &= \left(\lim_{m \to \infty} \varepsilon^{m\,(D_{\mathcal{W}}-3)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\left(X, \mathscr{P}_m, \mathscr{Q}_m\right) \tilde{\lambda}^p_{|f|,m,j,X} \, \widetilde{|f|}^p_{m,j,X} \right)^{\frac{1}{p}} . \end{aligned}$$

Besov Space on the Weierstrass Curve (Extension of the result given by Th. 6, p. 135, in[XV])

Given $k \in \mathbb{N}$, $k < \alpha \leq k+1$, $p \geq 1$ and $q \geq 1$, the *Besov space* $B^{p,q}_{\alpha}\left(\Gamma_{\mathcal{W}}\right)$ is defined as the set of functions $f \in L^p(\mu)$ such that there exists a sequence $\left(c_m\right)_{m\in\mathbb{N}} \in \ell^q$ of nonnegative real numbers such that for every $\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}$-net, one can find a spline function $spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right) \in \mathscr{P}ol_{[\alpha]}\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)$ satisfying, for all $m \in \mathbb{N}$,

$$\left\|f - spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right\|_{L^p(\mu)} \leq N_b^{(D_{\mathcal{W}}-3)\,m\,\alpha}\, c_m\,, \cdot \qquad \left(\mathscr{C}ond_{Besov\ spline}\right)$$

[XV] Alf Jonsson and Hans Wallin. *Function spaces on subsets of* $\mathbb{R}^n$. Mathematical reports (Chur, Switzerland). Harwood Academic Publishers, 1984.

Remark

The atomic decomposition used in[XVI] is obtained by introducing small neighborhoods of the curve under study (union of balls). Our polygonal domain appears to be a more natural choice. Indeed, unlike the aforementioned balls, the polygons involved do not overlap with each other, which works better for the required nets.

[XVI] M. Kabanava. "Besov Spaces on Nested Fractals by Piecewise Harmonic Functions". In: *Zeitschrift für Analysis und ihre Anwendungen* 31.2 (2012), pp. 183–201.

Besov Norm

Given $k \in \mathbb{N}$, $k < \alpha \leq k+1$, $p \geq 1$ and $q \geq 1$,we can define, as in[XVII], the $B_\alpha^{p,q}(\Gamma_{\mathscr{W}})$-*norm* of a function f defined on the Weierstrass Curve as

$$\|f\|_{B_\alpha^{p,q}(\Gamma_{\mathscr{W}})} = \|f\|_{L^p(\Gamma_{\mathscr{W}})} + \inf \left\{ \sum_{n \in \mathbb{N}} c_n^q \right\}^{\frac{1}{q}} ,$$

Yet, in order to obtain **a characterization** of the Besov space $B_\alpha^{p,q}(\Gamma_{\mathscr{W}})$ by means of its norm, it is more useful to deal with the equivalent norm given by

$$\|f\|_{B_\alpha^{p,q}(\Gamma_{\mathscr{W}})} = \|f\|_{L^p(\Gamma_{\mathscr{W}})} + \left\{ \iint_{(T,Y) \in \Gamma_{\mathscr{W}}^2} \frac{|f(T)-f(Y)|^q}{d_{eucl}^{D_{\mathscr{W}}+\alpha q}(T,Y)} \, d\mu^2 \right\}^{\frac{1}{q}} .$$

This enables one to make the link with discrete and fractal Laplacians, by means of the fractional difference quotients involved.

[XVII] Hans Wallin. "The trace to the boundary of Sobolev spaces on a snowflake". In: *manuscripta math.* 73 (1991), pp. 117–125.

Remark ~ *i*.

Characterizing Besov spaces on $\Gamma_{\mathscr{W}}$ by means of the previous norm is **directly associated to the definition of a sequence of (suitably renormalized) discrete graph Laplacians $(\Delta_m)_{m\in\mathbb{N}}$** on the sequence of prefractal approximations $\left(\Gamma_{\mathscr{W}_m}\right)_{m\in\mathbb{N}}$.
In a sense, it is also connected to the existence of the limit

$$\lim_{m\to\infty} \Delta_m$$

by means of **an equivalent pointwise formula expressed in terms of integrals**, somehow **the counterpart, in a way**, of the one which is well known in the case of the fractal Laplacian on the Sierpiński Gasket[XVIII],[XIX].

[XVIII] Jun Kigami. *Analysis on Fractals*. Cambridge University Press, 2001.

[XIX] Robert S. Strichartz. *Differential Equations on Fractals, A tutorial*. Princeton University Press, 2006.

Remark ~ *ii*.

The difficulty, in our context, is to obtain an equivalent formulation of the definition of Besov spaces with the sequence of discrete Laplacians alluded to in part *i*. Clearly, **a discrete Laplacian corresponds to the usual first difference**. Working with discrete Laplacians, along with atomic decompositions of functions, leads to expressions of the following form:

$$\lim_{m\to\infty} \varepsilon^{2\,m\,(D_{\mathscr{W}}-2)} \sum_{(T,Y)\in(\mathscr{P}_m \bigcup \mathscr{Q}_m)^2,\, Y \underset{m}{\sim} T} \mu^{\mathscr{L}}(T,\mathscr{P}_m,\mathscr{Q}_m)\,\mu^{\mathscr{L}}(Y,\mathscr{P}_m,\mathscr{Q}_m)\,\tilde{\lambda}_{f,m}\,\frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|^q}{d_{eucl}^{D_{\mathscr{W}}+(\alpha-k)\,q}(T,Y)}\,.$$

Theorem: Characterization of Besov Spaces[XX]

Given $k \in \mathbb{N}$, $k < \alpha \leq k+1$, $p \geq 1$ and $q \geq 1$, and a continuous function f given by means of an atomic decomposition of the form

$$f = \lim_{m \to \infty} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \tilde{\lambda}_{f,m,X} \, \tilde{f}_{m,X}$$

belongs to the Besov space $B^{p,q}_{\alpha}(\Gamma_{\mathscr{W}})$ if and only if the following two conditions are satisfied,

$$(3 - D_{\mathscr{W}}) \left\{ q \left(\frac{1}{p} - \frac{s-1}{D_{\mathscr{W}}} \right) \right\} + (2 - D_{\mathscr{W}})(D_{\mathscr{W}} + (\alpha - 1)\, q) < 2 \,, \quad (\mathscr{C}ond_{Besov})$$

and

$$\frac{D_{\mathscr{W}}}{3 - D_{\mathscr{W}}} + \frac{D_{\mathscr{W}}}{p} \leq s \,, \quad (\mathscr{C}ond_{L^p}) \cdot$$

[XX] Claire David and Michel L. Lapidus. *Iterated fractal drums ~ Some New Perspectives: Polyhedral Measures, Atomic Decompositions and Morse Theory*. 2022.

Trace of an $L^1_{loc}(\mathbb{R}^2)$ Function on the Weierstrass Curve

Along the lines of[XXI], page 15, or[XXII], we will say that an $L^1_{loc}(\mathbb{R}^2)$ function f is *strictly defined* at a vertex X of the Weierstrass Curve if the following limit exists and is given by

$$\bar{f}(X) = \lim_{m\to\infty} \frac{1}{\mu^{\mathscr{L}}\left(X, \mathscr{P}_m, \mathscr{Q}_m\right)} \sum_{Y \sim X} f(Y) < \infty \cdot$$

This enables us to define *the trace* $f_{|\Gamma_{\mathscr{W}}}$ of the function f on the Weierstrass Curve, via

$$\forall\, X \in \Gamma_{\mathscr{W}} \,:\, f_{|\Gamma_{\mathscr{W}}}(X) = \bar{f}(X) \cdot$$

The trace $\bar{f}$ of an $L^1_{loc}(\mathbb{R}^2)$ function thus naturally admits an atomic decomposition.

[XXI] Alf Jonsson and Hans Wallin. *Function Spaces on Subsets of R^n. Mathematical Reports, Vol. II, Part 1*. Harwood Academic Publishers, London, 1984.

[XXII] Hans Wallin. "The trace to the boundary of Sobolev spaces on a snowflake". In: *manuscripta math.* 73 (1991), pp. 117–125.

Associated Sobolev Space

We set

$$m_{\mathscr{W}} = \min_{t\in[0,1]} \mathscr{W}(t) \quad , \quad M_{\mathscr{W}} = \max_{t\in[0,1]} \mathscr{W}(t) \quad , \quad \Omega_{\mathscr{W}} = [0,1]\times[m_{\mathscr{W}}, M_{\mathscr{W}}]\cdot$$

Then,

$$\Gamma_{\mathscr{W}} \subset \Omega_{\mathscr{W}} \subset \mathbb{R}^2,$$

and, given $k \in \mathbb{N}$, and $p \geq 1$,

$$W_k^p\left(\mathring{\Omega}_{\mathscr{W}}\right) = \left\{f \in L^p\left(\mathring{\Omega}_{\mathscr{W}}\right), \forall\, \alpha \leq k,\, D^\alpha f \in L^p\left(\mathring{\Omega}_{\mathscr{W}}\right)\right\},$$

where $L^p\left(\mathring{\Omega}_{\mathscr{W}}\right)$ denotes the Lebesgue space of order p on $\mathring{\Omega}_{\mathscr{W}}$, while, for the multi-index $\alpha \leq k$, $D^\alpha f$ is the classical partial derivative of order α, interpreted in the weak sense.

Theorem: The Trace of Sobolev Spaces as Besov Spaces (counterpart of the corresponding one obtained in[XXIII], Chapter VI)

Given a positive integer k, and a real number $p \geq 1$, we set

$$\beta_{k,p} = k - \frac{2 - D_{\mathscr{W}}}{p} \cdot$$

We then have that

$$W_k^p \left(\mathring{\Omega}_{\mathscr{W}} \right)_{|\Gamma_{\mathscr{W}}} = B_\beta^{p,p} \left(\Gamma_{\mathscr{W}} \right) \cdot$$

[XXIII] Alf Jonsson and Hans Wallin. *Function spaces on subsets of* $\mathbb{R}^n$. Mathematical reports (Chur, Switzerland). Harwood Academic Publishers, 1984.

Corollary: Order of the Fractal Laplacian

In the case where $k = p = 2$, provided that

$$s > 1 + D_{\mathscr{W}} \, \frac{1 - D_{\mathscr{W}} + (2 - D_{\mathscr{W}})\,(2\,D_{\mathscr{W}} - 3)}{2\,(3 - D_{\mathscr{W}})} \, ,$$

we then have that

$$W_2^2\,(\Omega_{\mathscr{W}})_{|\Gamma_{\mathscr{W}}} = B^{2,2}_{\beta_{2,2}}\,(\Gamma_{\mathscr{W}}) \, ,$$

where

$$\beta_{2,2} = 2 - \frac{2 - D_{\mathscr{W}}}{2} = 2 - \frac{1}{2}\,\frac{\ln \lambda}{\ln N_b} > 2 \cdot$$

Consequently, by analogy with the classical theories, the Laplacian on the Weierstrass Curve arises as a differential operator of order $\beta_{2,2} \in \,]2, 3[$.

Connection with the Optimal Exponent of Hölder Continuity

We note that

$$\beta_{2,2} = 2 + \frac{\alpha_{\mathscr{W}}}{2},$$

where the Codimension $\alpha_{\mathscr{W}} = 2 - D_{\mathscr{W}} = -\frac{\ln \lambda}{\ln N_b} \in]0,1[$ is the best (i.e., optimal) Hölder exponent for the Weierstrass function, as was initially obtained by G. H. Hardy in[XXIV]), and then, by a completely different method – geometrically – in[XXV].

[XXIV] Godfrey Harold Hardy. "Weierstrass's Non-Differentiable Function". In: *Transactions of the American Mathematical Society* 17.3 (1916), pp. 301–325.

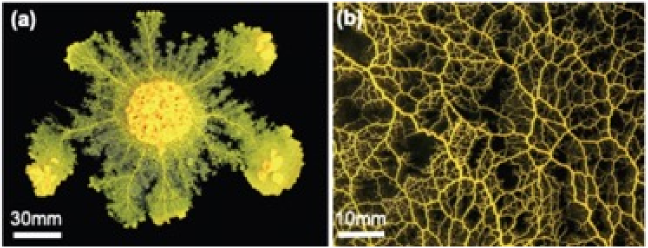
[XXV] Claire David and Michel L. Lapidus. *Weierstrass fractal drums - I - A glimpse of complex dimensions*. 2022.

The Polyhedral Measure In Real Life

The Polyhedral Measure In Real Life

⇝ Nature produces many **fractal-like structures**. Until now, the **tools of fractal geometry** have been little used to model the **morphogenesis** of these living forms.

⇝ The acellular model organism **Physarum polycephalum** grows in a **network and fractal branched way**.



(a) P. polycephalum plasmodium. (b) Vein network.
© A. Dussutour & C. Oettmeier.

⇝ The change of shape in **Physarum polycephalum** corresponds to a **change of fractal (complex) dimensions** (undergoing work with A. Dussutour, H. Henni, C. Godin).

⇝ Just as in our **mathematical theory**.

⇝ What is **the growth law**?

⇝ Can we find **the underlying variational principle**?

Forthcoming: The Magnitude

⇝ Counterpart of the **(topological) Euler characteristic**[XXVI].

⇝ New method for **numerically determining the Complex Dimensions of a fractal**[XXVII].

⇝ Also **connected** to **the polyhedral measure**.

[XXVI] Tom Leinster. "The magnitude of metric spaces". In: *Documenta Mathematica* 18 (2013), pp. 857–905. ISSN: 1431-0635.

[XXVII] Claire David and Michel L. Lapidus. *Fractal Complex Dimensions ~ A Bridge to Magnitude*. 2023.