A solution to fractal Hilbert's 16^{th} problem for slow-fast Liénard equations

Renato Huzak (Hasselt University, Belgium)

joint work with P. De Maesschalck, A. Janssens and G. Radunovic (FRABDYN HRZZ PZS 3055)

10.5.2023

Traditional Hilbert's 16^{th} problem (difficult) —–> Fractal Hilbert's 16^{th} problem

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Slow-fast Liénard equations

$$\dot{x} = y - F_{n+1}(x), \ \dot{y} = -\epsilon G_m(x)$$

Traditional Hilbert's 16^{th} problem

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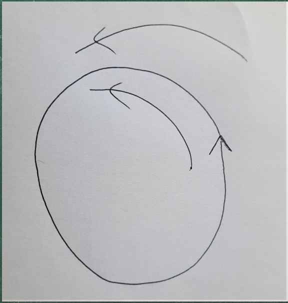
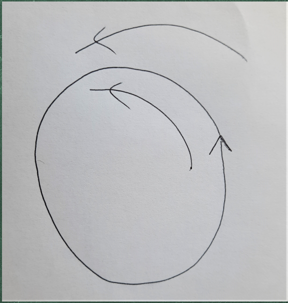
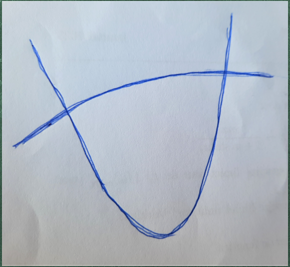
Is there a bound $H(n)$ on the number of limit cycles?

Is $H(n) \leq n^a$? (Smale)

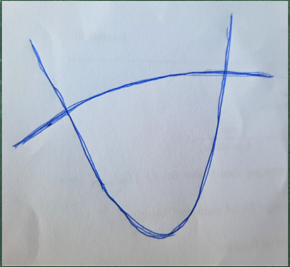




$H(2) < \infty$? Dumortier-Roussarie-Rousseau program, 1994



solve it locally in the (x, y)-phase space and parameter space



solve it locally in the (x, y)-phase space and parameter space

Our goal in slow-fast setting: the number of limit cycles —>
cardinality of the fractal spectrum

Slow-fast Liénard equations

$$\begin{cases} \dot{x} = y - \sum_{k=0}^{n+1} B_k x^k & A_m \neq 0 \\ \dot{y} = -\varepsilon \sum_{k=0}^{m} A_k x^k & B_{n+1} \neq 0 \end{cases}$$

$\epsilon \geq 0$ is the singular perturbation parameter kept small

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$m = 1$: classical Liénard equations of degree $n+1$

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$m > 1$: generalized Liénard equations

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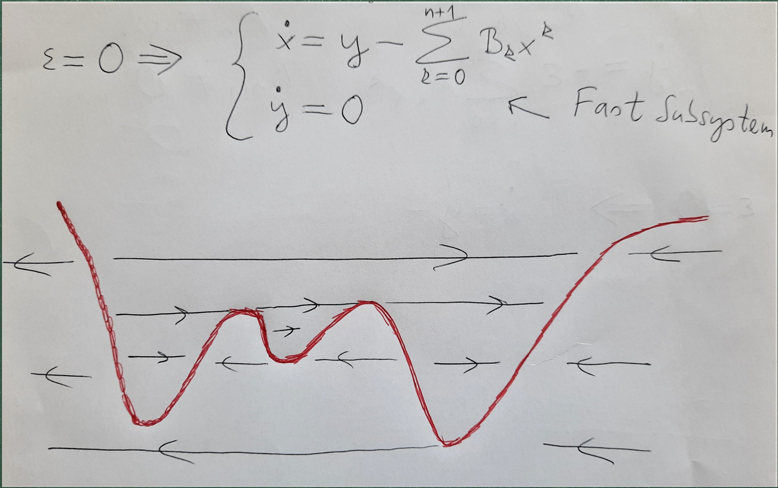
$m = 1$: classical Liénard equations of degree $n + 1$

$m > 1$: generalized Liénard equations

$n = 1$: Liénard equations with linear damping


$\varepsilon = 0 \Rightarrow \begin{cases} \dot{x} = y - \sum_{k=0}^{n+1} B_k x^k \\ \dot{y} = 0 \end{cases}$
← Fast Subsystem

$$\begin{cases} \dot{x} = y - \sum_{k=0}^{n+1} B_k x^k & A_m \neq 0 \\ \dot{y} = -\varepsilon \sum_{k=0}^{m} A_k x^k & B_{n+1} \neq 0 \end{cases}$$

$$\begin{cases} \dot{x} = y - \sum_{k=0}^{n+1} B_k x^k \\ \dot{y} = -\varepsilon \sum_{k=0}^{m} A_k x^k \end{cases} \qquad A_m \neq 0, \quad B_{n+1} \neq 0$$

$$\begin{cases} \varepsilon x' = y - \sum_{k=0}^{n+1} B_k x^k \\ y' = -\sum_{k=0}^{m} A_k x^k \end{cases} \overset{\varepsilon \to 0}{\Longrightarrow} \begin{cases} 0 = y - \sum_{k=0}^{n+1} B_k x^k \\ y' = -\sum_{k=0}^{m} A_k x^k \end{cases}$$

Slow Subsystem

$$\Rightarrow x' = -\frac{\sum_{k=0}^{m} A_k x^k}{\left(\sum_{k=0}^{n+1} B_k x^k\right)'}$$

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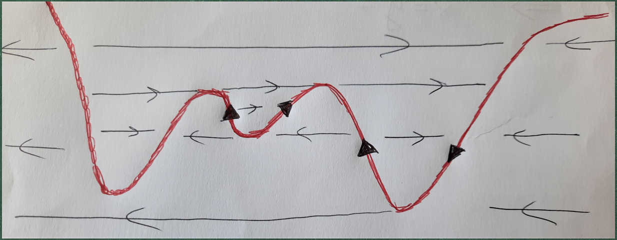
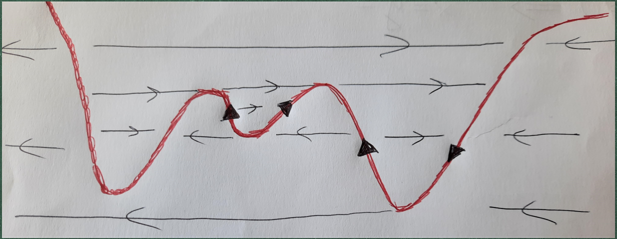
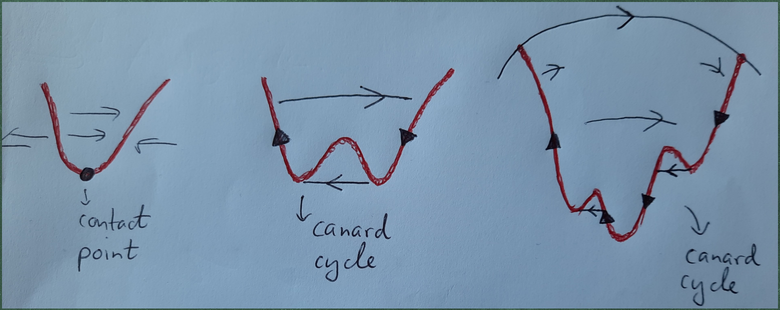
slow+fast



slow+fast



Limit periodic sets



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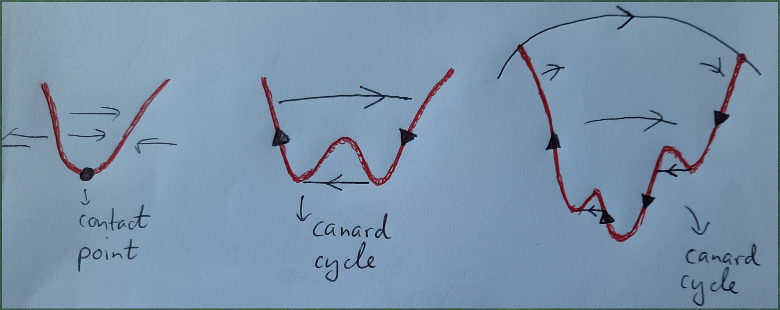
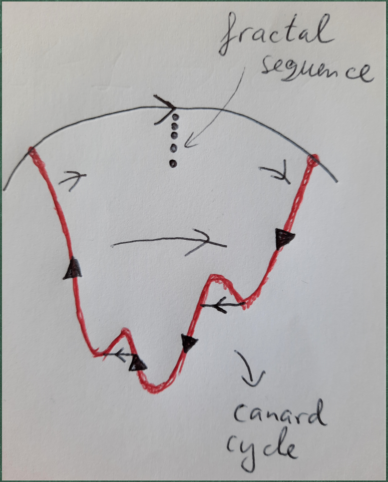
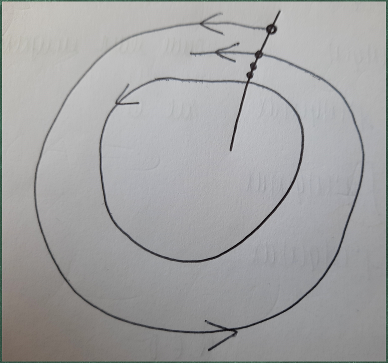


Canard cycles... [De Maesschalck,Dumortier,Roussarie,2021]

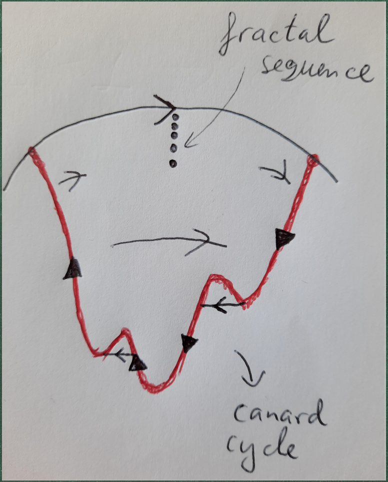
sequences near infinity



regular case [Zubrinic,Zupanovic,2005,2008]



to generate a sequence we use entry-exit (or slow-relation function)



Entry-Exit

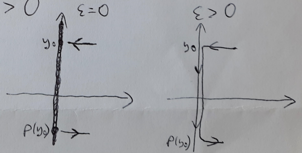
$$\begin{cases} \dot{x} = \bar{f}(x,y,\varepsilon)x^n \\ \dot{y} = \varepsilon \bar{g}(x,y,\varepsilon) \end{cases} \qquad n \geq 1$$

- $\bar{g}(0,y,0) < 0$
- $\bar{f}(0,y,0) > 0$ for $y < 0$

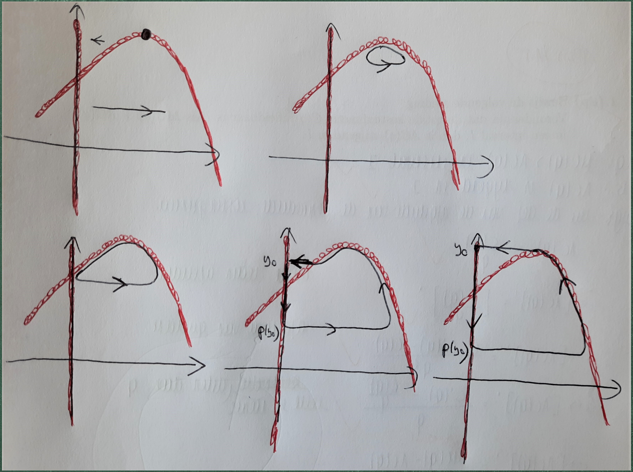
  $\bar{f}(0,y,0) < 0 \quad y > 0$

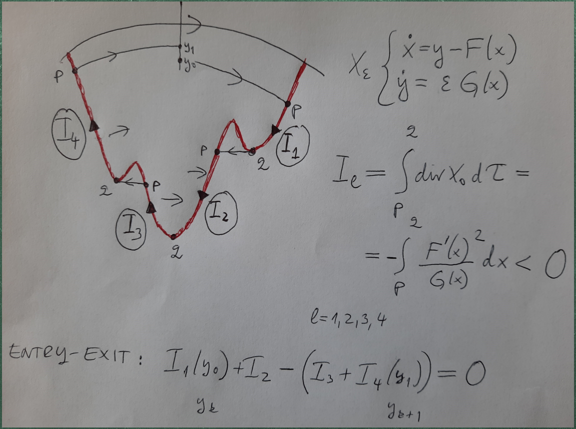
ENTRY-EXIT:

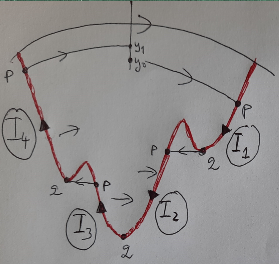
$$\int_{y_0}^{p(y_0)} \frac{\bar{f}(0,y,0)}{\bar{g}(0,y,0)}\,dy = 0$$



Predator-prey systems




$X_\varepsilon \begin{cases} \dot{x} = y - F(x) \\ \dot{y} = \varepsilon G(x) \end{cases}$
$I_\ell = \int_p^2 \operatorname{div} X_0 \, d\tau = -\int_p^2 \frac{F'(x)^2}{G(x)} dx < 0$
$\ell = 1, 2, 3, 4$
ENTRY-EXIT : $I_1(y_0) + I_2 - (I_3 + I_4(y_1)) = 0$
y_k
y_{k+1}
I_1
I_2
I_3
I_4
y_1
y_0



$$X_\varepsilon \begin{cases} \dot{x} = y - F(x) \\ \dot{y} = \varepsilon G(x) \end{cases}$$

$$I_\ell = \int_p^q \operatorname{div} X_0 \, d\tau = -\int_p^q \frac{F'(x)^2}{G(x)} dx < 0$$

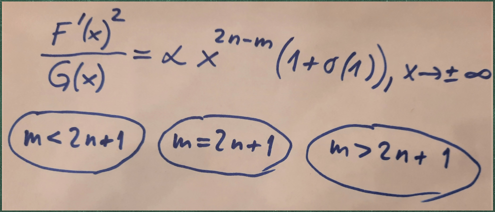
$\ell = 1, 2, 3, 4$

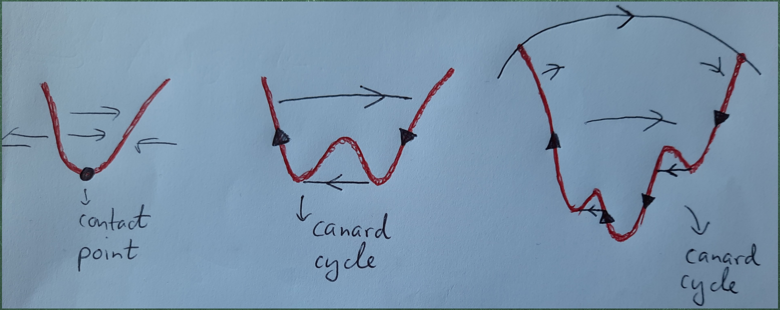
ENTRY-EXIT: $I_1(y_0) + I_2 - (I_3 + I_4(y_1)) = 0$

$y_k \qquad y_{k+1}$

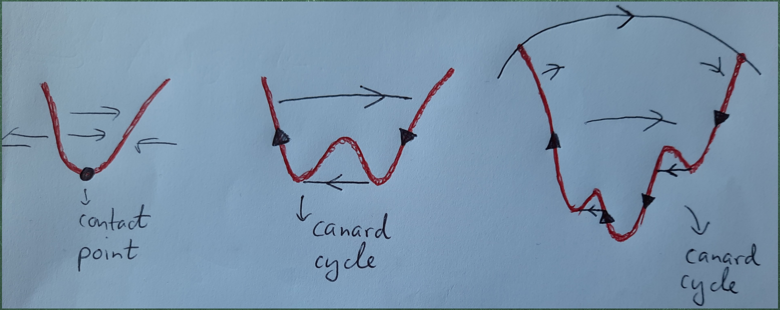
$I(y) \overset{def}{=} I_1(y) + I_2 - (I_3 + I_4(y)) \rightarrow$ slow div. integral

$I(y) \rightarrow I_*, 0, \pm\infty$ as $y \rightarrow +\infty$, $I_* \neq 0$


$\frac{F'(x)^2}{G(x)} = \alpha x^{2n-m}(1+o(1)), x \to \pm\infty$
$m < 2n+1$
$m = 2n+1$
$m > 2n+1$



[Huzak,2017] [Huzak,Vlah,2019] [Crnkovic,Huzak,Vlah,2021] [Huzak,Vlah,Zubrinic,Zupanovic,2023] [De Maesschalck,Huzak,Janssens,Radunovic,2023]



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Box dimension (see Falconer, Lapidus, Tricot, ...)

- Let $\delta > 0$ and $\delta \approx 0$
- $U(\delta) \overset{\text{def}}{=}$ the δ-neighborhood of a bounded $U \subseteq \mathbb{R}$
- $|U(\delta)| \overset{\text{def}}{=}$ the Leb. measure of $U(\delta)$

$\Rightarrow$ the lower box dimension:

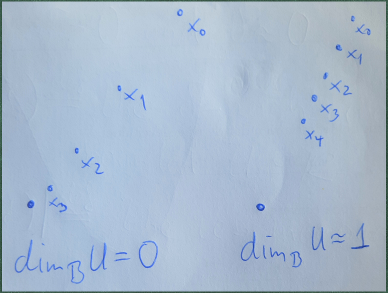
$$\underline{\dim}_B U = \liminf_{\delta \to 0} \left(1 - \frac{\ln |U(\delta)|}{\ln \delta}\right)$$

the upper box dimension:

$$\overline{\dim}_B U = \limsup_{\delta \to 0} \left(1 - \frac{\ln |U(\delta)|}{\ln \delta}\right)$$

$\dim_B U$

Box dimension measures the density of *U—the bigger the box dimension, the higher the density*



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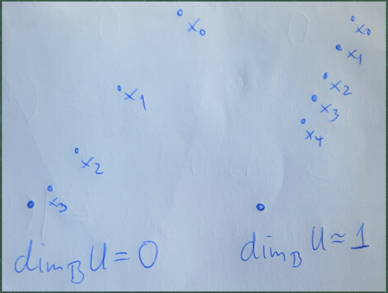

See [Box dimension of trajectories of 1-dim discrete dynamical systems, Elezovic,Zupanovic,Zubrinic,2007]

Slow-fast Liénard equations

$$
\begin{cases}
\dot{x} = y - \sum_{k=0}^{n+1} B_k x^k & A_m \neq 0 \\
\dot{y} = -\varepsilon \sum_{k=0}^{m} A_k x^k & B_{n+1} \neq 0
\end{cases}
$$

$m < 2n+1$ $\quad$ $m = 2n+1$ $\quad$ $m > 2n+1$

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$m < 2n+1 \qquad m = 2n+1 \qquad m > 2n+1$

$m \leq 2n+1$: Poincaré-Lyapunov disc of degree $(1, n+1)$

$$x = \bar{x}/r, y = 1/r^{n+1}$$

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$$x = \pm 1/r, y = \bar{y}/r^{\frac{m+1}{2}}$$

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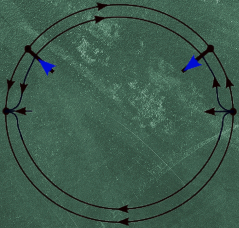
—a rescaling in x, y, t—

$$\begin{cases} \dot{x} = y - \left(x^{n+1} + \sum_{k=0}^{n} b_k x^k \right) \\ \dot{y} = -\varepsilon \left(A x^m + \sum_{k=0}^{m-1} a_k x^k \right) \end{cases}$$

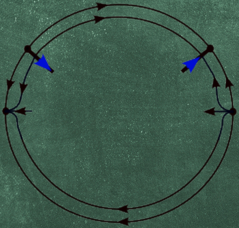
$A = 1$ if m is even and $m \neq 2n+1$

$A = \text{sign}(A_m)$ if m is odd and $m \neq 2n+1$

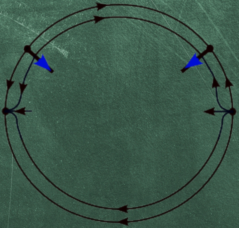
$A \neq 0$ if $m = 2n+1$



$A = 1$, m odd, n odd



$A = -1$, m odd, n odd



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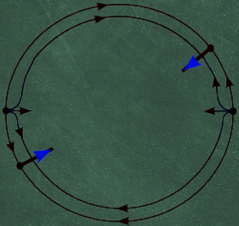

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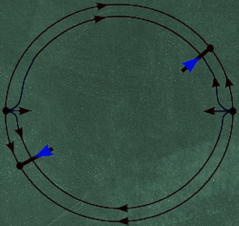

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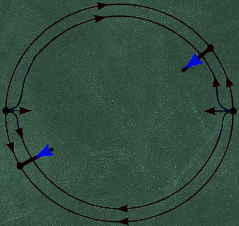
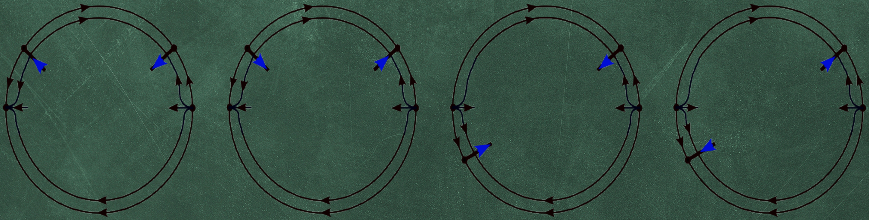


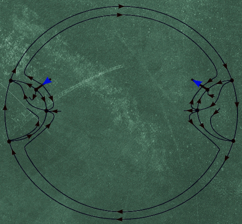
$A = 1$, m even, n even

$m = 2n + 1$

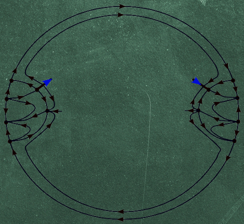


$A > 0,\ n$ odd $\qquad A < 0,\ n$ odd $\qquad A > 0,\ n$ even $\qquad A < 0,\ n$ even

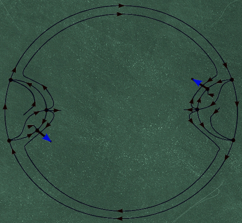
$m > 2n + 1$, m odd



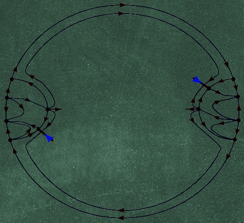
$A = 1$, n odd



$A = -1$, n odd

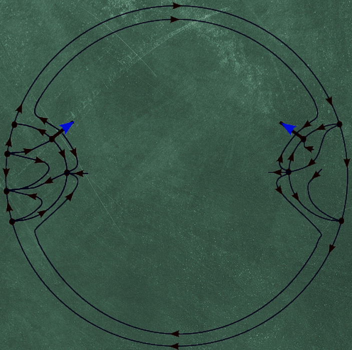


$A = 1$, n even



$A = -1$, n even

$m > 2n + 1$, m even



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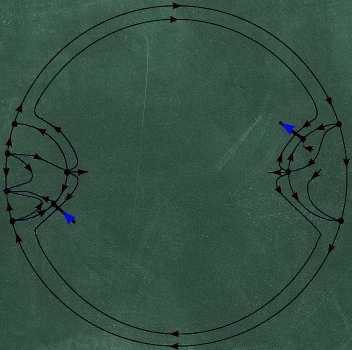

$A = 1$, n even

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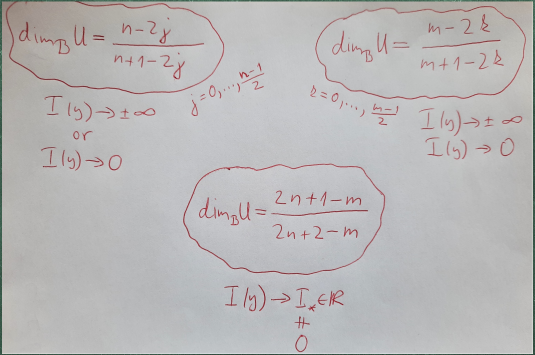
$A = 1$ if m is even and $m \neq 2n+1$

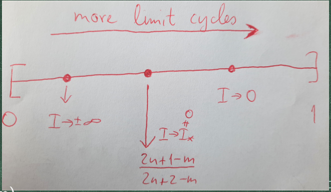
$A = \text{sign}(A_m)$ if m is odd and $m \neq 2n+1$

$A \neq 0$ if $m = 2n+1$

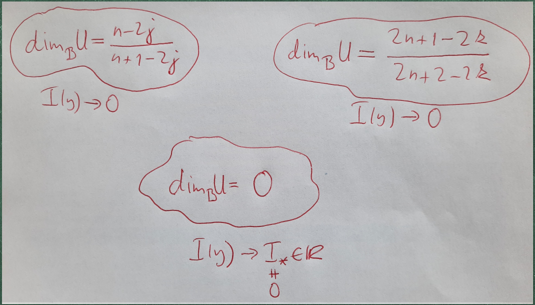
we break the symmetry: $b_{2j+1} = a_{2k} = 0$

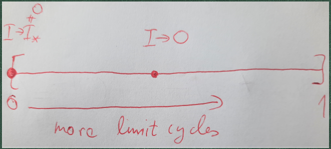
Theorem ($m < 2n+1$)





Theorem $(m = 2n + 1)$





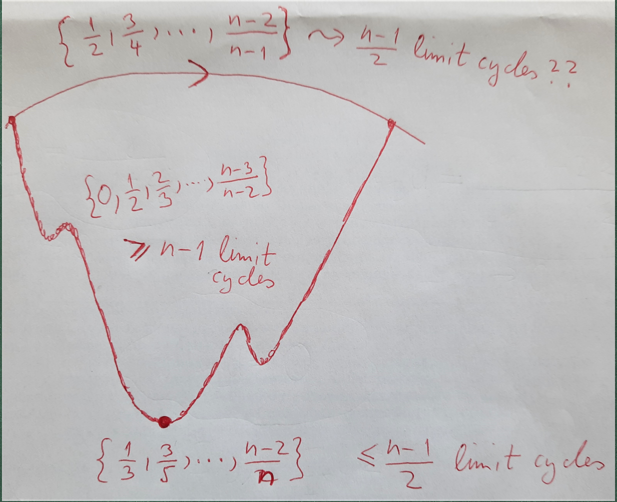
Theorem ($m > 2n+1$)

$I(y) \to 0$

$$\dim_B U = \frac{(n-2j)(m+1)}{(n-2j)(m+1)+2(n+1)}$$

$$\dim_B U = \frac{(m-2k)(m+1)}{(m-2k)(m+1)+2(n+1)}$$

Classical Liénard equations ($m < 2n+1$)



Thank you!