Normal forms for strongly hyperbolic logarithmic transseries and Dulac germs

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Contents

- Dulac germs, logarithmic transseries and normalization problem
- Formal solution of normalization problem in class of strongly hyperbolic logarithmic transseries
- Normalization of analytic maps with strongly hyperbolic logarithmic bounds
- Normalization of strongly hyperbolic Dulac germs

Dulac germs

Let $f : (0, d) \to \mathbb{R}$ be analytic on $(0, d)$, $d > 0$, such that

$$f \sim \lambda z^{\alpha} + \sum_{i \geq 1} z^{\alpha_i} P_i(-\log z), \qquad \longleftarrow \text{Dulac series}$$

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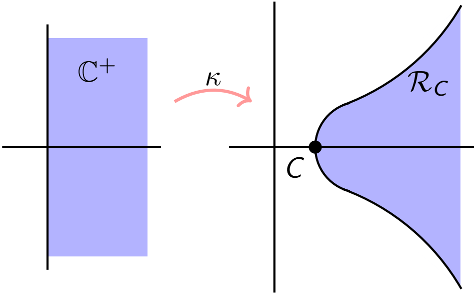
where:

- (α_i) is strictly increasing sequence of real numbers tending to $+\infty$ such that $\alpha_i > \alpha > 0$,
- P_i are polynomials with real coefficients and $\lambda > 0$.

Properties of Dulac germs

- Extension of Dulac germs to sufficiently large complex domains $\rightarrow$ standard quadratic domains



Slika: $\kappa(\mathbb{C}^+)$, where $\kappa(\zeta) = \zeta + C(\zeta + 1)^{\frac{1}{2}}$

- quasi-analyticity property $\longrightarrow$ Dulac series completely determines related Dulac germ

Normalization problem

- To solve the equation

$$\varphi(f(z)) = (\varphi(z))^{\alpha},$$

  for the fixed real $\alpha > 1$.

- f can be transseries, C^r-map, analytic map,...

- **The first goal:** solving the normalization equation in the class of logarithmic transseries

- **The second goal:** solving the normalization equation in the class of (complex) Dulac germs

Motivation

Theorem (Böttcher Theorem)

Let $f \in \mathrm{Diff}(\mathbb{C},0)$ *be a strongly hyperbolic complex analytic germ of diffeomorphism at zero, i.e.,* $f(z) = z^n + o(z^n)$, *for* $n \geq 2$. *There exists a parabolic analytic change of variables* $\varphi \in \mathrm{Diff}(\mathbb{C},0)$, $\varphi(z) = z + o(z)$, *such that* $\varphi(f(z)) = (\varphi(z))^n$.

Logarithmic transseries

- z is a formal variable at zero
- Let $\boldsymbol{\ell}_1 := -\frac{1}{\log z}$ and $\boldsymbol{\ell}_n := \boldsymbol{\ell}_{n-1} \circ \boldsymbol{\ell}_1$, $n \in \mathbb{N}_{\geq 1}$
- A logarithmic transseries f is a formal sum of type

$$f = \sum_{(\alpha, n_1, \ldots, n_k) \in \mathbb{R} \times \mathbb{Z}^k} a_{\alpha, n_1, \ldots, n_k} \cdot z^{\alpha} \boldsymbol{\ell}_1^{n_1} \cdots \boldsymbol{\ell}_k^{n_k} = \sum_{\alpha \in \mathbb{R}} z^{\alpha} R_{\alpha}$$

with well-ordered support (lexicographic order on $\mathbb{R} \times \mathbb{Z}^k$) whose minimum is strictly bigger than $(0, 0, \ldots, 0)$.

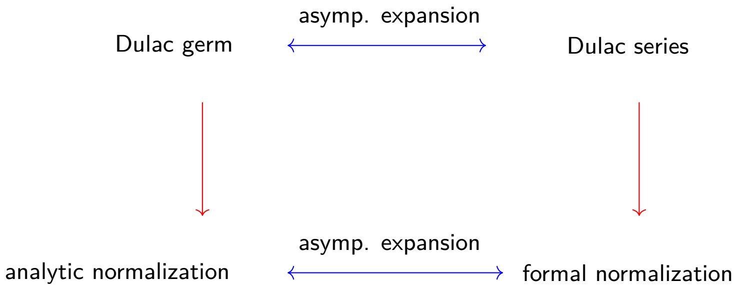
Composition

- Main goal for the definition of the composition $\longrightarrow \mathcal{L}_k$, $k \in \mathbb{N}$, should be closed for the composition
- $\mathcal{L}_k^H$ - set of all transseries $f = \lambda z^\alpha + \mathrm{h.o.t.}$, $\alpha > 0$, $\lambda > 0$
- We say that $f \in \mathcal{L}_k^H$ is:
  - parabolic if $f = z + \mathrm{h.o.t.}$,
  - hyperbolic if $f = \lambda z + \mathrm{h.o.t.}$, for $\lambda > 0$, $\lambda \neq 1$,
  - strongly hyperbolic if $f = \lambda z^\alpha + \mathrm{h.o.t.}$, for $\lambda > 0$, $\alpha > 0$, $\alpha \neq 1$.

- Formal Taylor theorem $\longrightarrow$ well-defined formal composition on the set $\mathcal{L}_k^H$:

$$f \circ (\lambda z^\alpha + g) := f(\lambda z^\alpha) + \sum_{i \geq 1} \frac{f^{(i)}(\lambda z^\alpha)}{i!} g^i$$

Three steps for solving normalization equation for Dulac germs



Three steps for solving normalization equation for Dulac germs

- finding formal solution in the class of Dulac series
  - formal solution in larger class of logarithmic transseries

Three steps for solving normalization equation for Dulac germs

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- finding analytic solution for Dulac germs
  - finding analytic solution in the larger class of analytic maps with strongly hyperbolic logarithmic bounds

Three steps for solving normalization equation for Dulac germs

- finding formal solution in the class of Dulac series
  - formal solution in larger class of logarithmic transseries
- finding analytic solution for Dulac germs
  - finding analytic solution in the larger class of analytic maps with strongly hyperbolic logarithmic bounds
- connection of formal solution and the analytic solution via certain homological equation

Step 1: formal solution in the class of Dulac series

Normalization theorem for strongly hyperbolic logarithmic transseries

Theorem

Let $f \in \mathcal{L}_k^H$, $f = z^\alpha + \mathrm{h.o.t.}$, $\alpha \in \mathbb{R}_{>0}$, $\alpha \neq 1$, be a strongly hyperbolic logarithmic transseries. Then:

- *There exists a unique solution $\varphi \in \mathcal{L}_k^0$ of the normalization equation:*

$$\varphi \circ f \circ \varphi^{-1} = z^\alpha.$$

- *If $\alpha > 1$, then, for every initial condition $h \in \mathcal{L}_k^0$, the Böttcher sequence*

$$\left(z^{\frac{1}{\alpha^n}} \circ h \circ f^{\circ n} \right)_n$$

converges to the normalization φ in the weak topology on $\mathcal{L}_k^0$ as n tends to $+\infty$.

Normalization theorem for strongly hyperbolic Dulac series

Theorem

Let $f = z^{\alpha} + \mathrm{h.o.t.}$, $\alpha \in \mathbb{R}_{>0}$, $\alpha \neq 1$, *be a strongly hyperbolic Dulac series. Then there exists a unique parabolic Dulac series* φ *such that*

$$\varphi \circ f \circ \varphi^{-1} = z^{\alpha}.$$

Step 2: finding analytic solution for Dulac germs

Normalization of analytic maps with strongly hyperbolic logarithmic bounds

Suppose that map f defined on domain $D \subseteq \mathbb{C}^+$ has the following asymptotic bound:

$$f(\zeta) = \alpha\zeta + o(\boldsymbol{L}_k^{-\varepsilon}), \quad \text{as } \Re(\zeta) \to +\infty \text{ uniformly on } D_C.$$

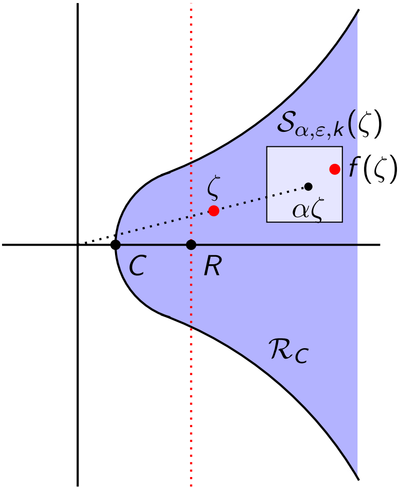
for $\alpha > 1$, $\varepsilon > 0$, $k \in \mathbb{N}$.
Here, $\boldsymbol{L}_1, \ldots \boldsymbol{L}_k$ are iterated logarithms.

Question: Can we find some sufficient conditions on D such that there exists $R > 0$ such that

$$D_R := D \cap ([R, +\infty) \times \mathbb{R})$$

is f-invariant ?

Solution: domains of type (α, ε, k)


$\mathcal{S}_{\alpha,\varepsilon,k}(\zeta)$
$f(\zeta)$
ζ
$\alpha\zeta$
C
R
$\mathcal{R}_C$

Proposition

Let $\alpha \in \mathbb{R}_{>1}$, $\varepsilon \in \mathbb{R}_{>0}$ *and* $k \in \mathbb{N}$. *Let* $D \subseteq \mathbb{C}$ *be a domain of type* (α, ε, k) *and let* $f : D_C \to \mathbb{C}$, $C > \exp^{\circ k}(0)$, *be an analytic map with the following asymptotic behaviour:*

$$f(\zeta) = \alpha\zeta + o(\boldsymbol{L}_k^{-\varepsilon}), \quad \text{as } \Re(\zeta) \to +\infty \text{ uniformly on } D_C.$$

Here,

$$\boldsymbol{L}_1 := \log(\zeta), \ldots, \boldsymbol{L}_k := \log(\boldsymbol{L}_{k-1}),$$

where log *represents the principal branch of the logarithm. Then, for every* $R > C$ *sufficiently large, the domain* D_R *is* f*-invariant.*

Linearization theorem for maps on admissible domains

Theorem

Let $\alpha \in \mathbb{R}_{>1}$, $\varepsilon \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$. Let $D \subseteq \mathbb{C}^+$ be a domain of type (α, ε, k). For $C > \exp^{\circ k}(0)$, let $f : D_C \to \mathbb{C}$ be an analytic map such that

$$f(\zeta) = \alpha\zeta + o(\boldsymbol{L}_k^{-\varepsilon}), \quad \text{as } \Re(\zeta) \to +\infty \text{ uniformly on } D_C.$$

Then:

- *(Existence) For a sufficiently large $R > \exp^{\circ k}(0)$ there exists an analytic normalizing map φ on the f-invariant subdomain $D_R \subseteq D$. That is, φ satisfies*

  $$(\varphi \circ f)(\zeta) = \alpha \cdot \varphi(\zeta), \text{ for all } \zeta \in D_R^f.$$

  Moreover, φ is the uniform limit on D_R of the Böttcher sequence $\left(\frac{1}{\alpha^n} f^{\circ n}\right)_n$ in the ζ-chart.

- (Asymptotics) *The normalization φ is tangent to identity, i.e., $\varphi(\zeta) = \zeta + o(1)$, uniformly on $D_R \subseteq \mathbb{C}^+$, as $\Re(\zeta) \to +\infty$.*

Step 3: connection of formal solution and the analytic solution via certain homological equation

Normalization of strongly hyperbolic (complex) Dulac germs

Theorem

Let f be a strongly hyperbolic complex Dulac germ and let $\widehat{f}(\zeta) = \alpha\zeta + o(1)$, $\alpha \in \mathbb{R}_{>1}$, be its asymptotic expansion in the ζ-chart. Then:

- *There exists the unique parabolic complex Dulac germ φ (given in the ζ-chart) which is a solution of the normalization equation:*

$$\varphi \circ f = \alpha \cdot \varphi.$$

  Furthermore, if f is a real Dulac germ, so is φ.
- *$\varphi \sim \widehat{\varphi}$, uniformly as $\Re(\zeta) \to +\infty$, where $\widehat{\varphi}(\zeta)$ is the unique solution of the formal normalization equation.*

Steps of the proof:

- Transition to the logarithmic chart $\zeta := -\log z$:

$$(\varphi \circ f)(z) = (\varphi(z))^{\alpha} \iff (\varphi \circ f)(\zeta) = \alpha \cdot \varphi(\zeta)$$

$$\widehat{f}(z) = z^{\alpha} + \sum_{i \geq 1} z^{\alpha_i} P_i(-\log z) \iff \widehat{f}(\zeta) = \alpha \cdot \zeta + \sum_{i \geq 1} \exp(-\zeta \beta_i) Q_i(\zeta)$$

- Suppose that

$$\widehat{\varphi} = \zeta + \sum_{n=1}^{+\infty} \exp(-\zeta \beta_n) R_n(\zeta)$$

Then write $\varphi_0 := \zeta$ and

$$\varphi_n := \zeta + \sum_{i=1}^{n} \exp(-\zeta \beta_i) R_i(\zeta)$$

Note that $\widehat{\varphi} = \sum_{n \in \mathbb{N}} \varphi_n$.

Sketch of the proof:

- $$((\varphi - \varphi_n) \circ f)(\zeta) - \alpha \cdot (\varphi - \varphi_n)(\zeta) = o\left(\mathrm{e}^{-(\beta_n + \nu_n)\zeta}\right),$$

- Solving the homological equation $\psi \circ f(\zeta) - \alpha \cdot \psi(\zeta) = h = o(\mathrm{e}^{-\nu\zeta})$, $\nu > 0$, on a standard quadratic domain, uniformly as $\mathrm{Re}\,\zeta \to +\infty$

$$\psi(\zeta) := -\sum_{n=0}^{+\infty} \frac{1}{\alpha^{n+1}} \cdot h\left(f^{\circ n}(\zeta)\right)$$

- Estimation and uniqueness of the solution ψ:

$$\psi(\zeta) = O(\mathrm{e}^{-\nu\zeta}),$$

uniformly as $\Re(\zeta) \to +\infty$.

- $$\to \quad \psi = \varphi - \varphi_n \quad \to \quad (\varphi - \varphi_n)(\zeta) = O\left(\mathrm{e}^{-(\beta_n + \nu_n)}\right).$$

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Thank you for your attention!