Probability of existence of limit cycles for a family of planar systems

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- B. Coll, A. Gasull and R. P., *Probability of existence of limit cycles for a family of planar systems*. Jour. Diff. Eq. 2023 (preprint)

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Introduction

We are interested in knowing the probability of existence of limit cycles for the planar family

$$\dot{x} = Af(x) + Bg(y),$$
$$\dot{y} = Cf(x) + Dg(y),$$

where f and g smooth functions satisfying $f(0) = g(0) = 0$.

It seems plausible to require that the real random variables A, B, C, D be independent identically distributed (iid) and continuous.

We will focus on the case $g(y) \equiv y$.

Distribution and probability space of the random variables

Consider

$$\dot{x} = Af(x) + Bg(y), \quad \dot{y} = Cf(x) + Dg(y),$$

where the random variables A, B, C, D are iid continuous random variables

$A, B, C, D \sim N(0, \sigma^2)$
density function $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}$ for each one.

Give rise to a random vector field (A, B, C, D) with a uniform distribution on $\mathbb{S}^3$:

$(\Omega, \mathcal{F}, P)$ with sample space $\Omega = R^4$, $\mathcal{F}$ the σ-algebra generated by the open sets of R^4 and $P : \mathcal{F} \to [0,1]$ is the probability function with joint density function $f(a,b,c,d) = \frac{1}{4\pi^2} e^{-(a^2+b^2+c^2+d^2)/2}$, where for simplicity we take variance one in each marginal density function.

Obtaining probabilities

If we want to obtain the phase portrait probability of having a saddle, for instance, for the linear random differential equation

$$\dot{x} = Ax + By, \quad \dot{y} = Cx + Dy,$$

we have to compute

$$P(AD - BC < 0) = \frac{1}{4\pi^2} \int_{\mathcal{U}} \mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}} \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}d,$$

where $\mathcal{U} := \{(a, b, c, d) \in \mathbb{R}^4 : ad - bc < 0\}$.

In general, depending on $\mathcal{U}$, this integral can be calculated analytically or, if not, approximated using Monte Carlo method.

Since the probability density function is positive, any non-empty event described by algebraic inequalities is measurable and has positive probability; whereas, measurable events such that their description is by a non-trivial algebraic equality have zero probability.

On planar random systems using a similar approach:

A. Cima, A. Gasull, V. Mañosa. *Phase portraits of random planar homogeneous vector fiels.* Qual. Theory Dyn. Syst. (2021)

A. Cima, A. Gasull, V. Mañosa. *Stability index of linear random dynamical systems.* Elec. J. of Qual. Theory of Differential Equations, Paper No. 15 (2021)

B. Coll, A. Gasull, R. P. *Probability of occurrence of some planar random quasi-homogeneous vector fields*. Mediterranean Journal of Mathematics (2022)

B.K. Pagnoncelli, H. Lopes and C.F.B. Palmeira. *Sampling linear ODE.* (in Portuguese). Matemática Universitária (2009)

Statements of the main results in the deterministic case

Theorem 1

Consider system

$$\dot{x} = af(x) + bg(y), \quad \dot{y} = cf(x) + dg(y),$$

with f and g smooth, such that $f(0) = g(0) = 0$. Then:

(i) If $abcd \leq 0$, it has not limit cycles.

(ii) Assume that f and g are analytic, $f(x) = x^{2l-1} + O(x^{2l})$ and $g(y) = y^{2k-1} + O(y^{2k})$, with $k \neq l$. Then, there exist a, b, c, d such that it has at least one limit cycle surrounding the origin, which whenever it exists is hyperbolic.

(iii) There exist f and g such that for some values of a, b, c and d, the system has more than one limit cycle surrounding the origin. Moreover, the same also holds with $g(y) \equiv y$.

case $\mathbf{g(y) \equiv y}$:

Theorem 2

Consider system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

with $ad \neq 0$. Let $f(x)$ be the polynomial $f(x) = \alpha x^k + \sum_{k<i<m} f_i x^i + \beta x^m$, with $\alpha\beta \neq 0$, $k \leq m$ odd integers and $m > 1$. Assume moreover that $x = 0$ is the unique real root of $f(x) = 0$.

(i) If $\beta(ad - bc) \leq 0$ then it has no periodic orbits.

(ii) If $\beta(ad - bc) > 0$ and, either $k = 1$ and $\beta a(a\alpha + d) > 0$, or $k > 1$ and $\beta ad > 0$, then it has zero or an even number of limit cycles.

(iii) If $\beta(ad - bc) > 0$ and, either $k = 1$ and $\beta a(a\alpha + d) < 0$, or $k > 1$ and $\beta ad < 0$, then it has an odd number of limit cycles.

In all the cases, each limit cycle is counted with its own multiplicity.

Theorem 3

Consider system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

where f is smooth and $f(0) = 0$. Assume that

$$M(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x)$$

does not change sign and vanishes at isolated points, where $F' = f$ and $F(0) = 0$. Let K be the number of bounded intervals (counting also intervals degenerated to a point as intervals) of the closed set

$$\{x \in \mathbb{R} : \Delta(x) = (af(x) + dx)^2 - 8(ad - bc)F(x) \geq 0\}.$$

Then the system has at most K limit cycles, all of them hyperbolic.

Corollary 4

Consider system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

and assume that

$$M(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x)$$

does not change sign and vanishes at isolated points. Assume also that the origin is the only equilibrium point of the system. Then it has at most one limit cycle, and when it exists, it is hyperbolic.

Corollary 5

Consider system

$$\dot{x} = ax^{2n-1} + by, \quad \dot{y} = cx^{2n-1} + dy,$$

where $n > 1$ is an integer. It has at most one limit cycle and, when it exists, it is hyperbolic. Moreover, it exists if and only if $ad - bc > 0$ and $ad < 0$ and its stability is given by the sign of $-d$.

Proposition 6

System

$$\dot{x} = a(\alpha x + x^3) + by, \quad \dot{y} = c(\alpha x + x^3) + dy,$$

with $\alpha \geq 0$ has at most one limit cycle. Moreover the limit cycle exists if and only if $ad - bc > 0$ and $a(a\alpha + d) < 0$ and its stability is given by the sign of $-(a\alpha + d)$.

Theorem (part 1)

Consider random system

$$\dot{x} = Af(x) + By, \quad \dot{y} = Cf(x) + Dy,$$

where $f(x) = \alpha x^k + \sum_{k<i<m} f_i x^i + \beta x^m$, with $\alpha\beta \neq 0$, $k \leq m$ odd integers, $m > 1$, and A, B, C, D iid random variables with distribution $N(0, 1)$. Assume also that $x = 0$ is the unique real root of $f(x) = 0$. Then

(i) When $k > 1$ the probability of having an odd number of limit cycles is $1/8$, and the probability of not having limit cycles or to have an even number is $7/8$.

(ii) When $k = 1$ and $\beta > 0$, the probability of having an odd number of limit cycles is $P^+(\alpha) \leq 1/2$, and the probability of not having limit cycles or to have an even number is $1 - P^+(\alpha)$.

(iii) When $k = 1$ and $\beta < 0$, the same results that in item (ii) hold but changing P^+ by P^-, where $P^-(\alpha) = P^+(-\alpha)$.

In all the cases, each limit cycle is counted with its own multiplicity.

Theorem (part 2)

The function $P^+ : \mathbb{R} \to (0, 1/2)$ is

$$P^+(\alpha) = \frac{1}{4\pi^2} \iiiint_{T(\alpha)} \mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}} \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}d,$$

where $T(\alpha) = \{(a, b, c, d) \, : \, ad - bc > 0, \, a(a\alpha + d) < 0\}$, a decreasing function that satisfies

$$\lim_{\alpha \to -\infty} P^+(\alpha) = 1/2, \quad P^+(0) = 1/8, \quad \lim_{\alpha \to +\infty} P^+(\alpha) = 0,$$

α	-100	-10	-1	0	1	10	100
MC-10^6	0.4984	0.4814	0.3127	0.1255	0.0624	0.0128	0.0016
MC-10^8	0.49829	0.48129	0.31247	0.12498	0.06254	0.01303	0.00155

Table: Some approximated values of $P^+(\alpha)$ obtained by Monte Carlo (MC) simulation taking 10^6 and 10^8 random systems. We know that $P^+(0) = 1/8 = 0.125$.

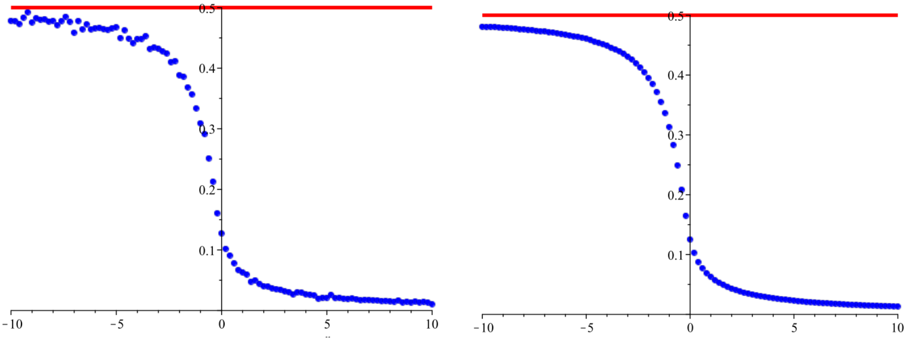

Figure: Numerical approximation of $P^+(\alpha)$ using Monte Carlo simulation with samples of $N = 10^4$ (left) and $N = 10^6$ (right) points for 101 equidistributed values of α in $[-10, 10]$. Expected error of order 10^{-3} and 10^{-4}, resp.

Samples of the random vector (A, B, C, D) checking how many of them, J, satisfy $AD - BC > 0$ and $A(\alpha A + D) < 0$. Then $P^+(\alpha) \approx J/N$.

Due to the law of large numbers and the law of iterated logarithm, this approach gives an absolute error of order $O(((\log\log N)/N)^{1/2})$.

Corollary

Consider random system

$$\dot{x} = Ax^k + By, \quad \dot{y} = Cx^k + Dy,$$

where $k > 1$ is an odd integer and A, B, C, D are iid random variables with distribution $N(0, 1)$. Then:

- the probability of having one limit cycle is $1/8$, and
- the probability of not having limit cycles is $7/8$.

Proposition

Consider random system

$$\dot{x} = A(\alpha x + x^3) + By, \quad \dot{y} = C(\alpha x + x^3) + Dy,$$

with $\alpha > 0$ and A, B, C, D iid random variables with distribution $N(0,1)$. Then, for each $\varepsilon > 0$ there exists α big enough such that it has limit cycles with a positive probability, smaller that ε. Moreover, when the limit cycle exists it is unique.

Theorem 1.

Consider system

$$\dot{x} = af(x) + bg(y), \quad \dot{y} = cf(x) + dg(y),$$

with f and g smooth, such that $f(0) = g(0) = 0$. Then:

(i) If $abcd \leq 0$, it has not limit cycles.

(ii) Assume that f and g are analytic, $f(x) = x^{2l-1} + O(x^{2l})$ and $g(y) = y^{2k-1} + O(y^{2k})$, with $k \neq l$. Then, there exist a, b, c, d such that it has at least one limit cycle surrounding the origin, which whenever it exists is hyperbolic.

(iii) There exist f and g such that for some values of a, b, c and d, the system has more than one limit cycle surrounding the origin. Moreover, the same also holds with $g(y) \equiv y$.

Proof of item (i): Let F and G be such that $F'(x) = f(x)$ and $G'(y) = g(y)$. Taking $H(x,y) = cF(x) - bG(y)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}H(x,y) = \dot{H}(x,y) = cf(x)\dot{x} - bg(y)\dot{y} = acf^2(x) - bdg^2(y).$$

If $abcd < 0 \Rightarrow H$ is a Lyapunov function.
Proof of item (ii): We prove that for ε small enough, system

$$\dot{x} = \varepsilon\alpha f(x) + g(y), \quad \dot{y} = -f(x) + \varepsilon\delta g(y),$$

has at least one limit cycle by studying the corresponding Abelian integral:

$$\begin{aligned} I(h) &= -\delta \int_{\gamma_h} g(y)\mathrm{d}x + \alpha \int_{\gamma_h} f(x)\mathrm{d}y \\ &= \delta \iint_{\mathrm{Int}(\gamma_h)} g'(y)\mathrm{d}x\,\mathrm{d}y + \alpha \iint_{\mathrm{Int}(\gamma_h)} f'(x)\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

where γ_h is the oval of the level curve $H(x,y) = F(x) + G(y) = h$ surrounding the origin and $\mathrm{Int}(\gamma_h)$ denotes the region surrounded by γ_h.

Continuation of the proof of item (ii): The key point is that

$$I(h) = \delta \iint_{\mathrm{Int}(\gamma_h)} g'(y)\mathrm{d}x\,\mathrm{d}y + \alpha \iint_{\mathrm{Int}(\gamma_h)} f'(x)\mathrm{d}x\,\mathrm{d}y,$$

and

$$\iint_{\mathrm{Int}(\gamma_h)} f'(x)\mathrm{d}x\,\mathrm{d}y \sim C_1\, h^{1+(l-k)/(2kl)},$$
$$\iint_{\mathrm{Int}(\gamma_h)} g'(y)\mathrm{d}x\,\mathrm{d}y \sim C_2\, h^{1+(k-l)/(2kl)},$$

in a neighborhood of $h = 0$ for $h > 0$. As a consequence, near $h = 0$,

$$I(h) \sim K_1 h^{1+(l-k)/(2kl)} + K_2 h^{1+(k-l)/(2kl)},$$

when $k \neq l$, it has at most a simple zero (ECT-system) and there are combinations for which it exists.

Proof of item (iii): This is a consequence of the following proposition that is proved by computing Lyapunov constants, $L_j, j = 0, 1, 2, \dots$

Proposition

(i) There are systems with f a polynomial of degree 4 and $g(y) \equiv y$, having at least 3 hyperbolic limit cycles surrounding the origin.
(ii) There are systems with f and g polynomials of degree 3, having at least 3 hyperbolic limit cycles surrounding the origin.

The only computational difficulty is that our system with a weak focus at the origin writes as

$$\dot{x} = y + rx + P(x, y), \quad \dot{y} = -x - ry + Q(x, y), \quad |r| < 1,$$

instead of the usual canonical form. We obtain then Lyapunov constants by looking for a Lyapunov function of the form (sugg. by Torregrosa)

$$H(x, y) = \sum_{k \geq 2} H_k(x, y), \quad \text{with} \quad H_2(x, y) = x^2 + y^2 + 2rxy.$$

Theorem 3

Consider system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

where f is smooth and $f(0) = 0$. Assume that

$$M(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x)$$

does not change sign and vanishes at isolated points, where $F' = f$ and $F(0) = 0$. Let K be the number of bounded intervals (counting also intervals degenerated to a point as intervals) of the closed set

$$\{x \in \mathbb{R} : \Delta(x) = (af(x) + dx)^2 - 8(ad - bc)F(x) \geq 0\}.$$

Then the system has at most K limit cycles, all of them hyperbolic.

Idea of the proof:

We follow the ideas developed by Gasull and Giacomini to look for a suitable Dulac function, $1/V$, to apply Bendixson-Dulac theorem to our system in a suitable region. We start with the well known formula

$$\operatorname{div}\left(\frac{P}{V},\frac{Q}{V}\right)=\frac{V\operatorname{div}(P,Q)-V_xP-V_yQ}{V^2}=:\frac{R}{V^2}.$$

The first key idea is to search for a function

$$V(x,y)=y^2+v(x)y+w(x),$$

for some v and w, such that when we compute R we obtain

$$R(x,y)=\frac{(ad-bc)}{b^2}M(x),$$

and since by hypothesis M does not change sign and vanishes only at isolated points, we can apply Bendixson-Dulac Theorem and the maximum number of limit cycles is given by the number of holes of $\mathbb{R}^2\setminus\{V(x,y)=0\}$.

Since V is quadratic on y, this number is controlled by the number of intervals of the discriminant of $V(x,y)$ with respect to y, that is the $\Delta(x)$ of the statement.

Corollary 4

Consider system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

and assume that

$$M(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x)$$

does not change sign and vanishes at isolated points. Assume also that the origin is the only equilibrium point of the system. Then it has at most one limit cycle, and when it exists, it is hyperbolic.

In this case, the uniqueness of the critical point implies that $\mathbb{R}^2 \setminus \{V(x,y) = 0\}$ has at most 1 hole $\Rightarrow$ [Theorem 3] at most one limit cycle.

Deterministic systems: uniqueness of the limit cycle

Corollary

Consider system

$$\dot{x} = ax^{2n-1} + by, \quad \dot{y} = cx^{2n-1} + dy,$$

where $n > 1$ is an integer. It has at most one limit cycle and, when it exists, it is hyperbolic. Moreover, it exists if and only if $ad - bc > 0$ and $ad < 0$ and its stability is given by the sign of $-d$.

When $ad > 0$, the classical Dulac criterion applies since the divergence of the vector field does not change sign.
When $ad < 0$, from Theorem 3, $K = 1$.

Proposition 6

System

$$\dot{x} = a(\alpha x + x^3) + by, \quad \dot{y} = c(\alpha x + x^3) + dy,$$

with $\alpha \geq 0$ has at most one limit cycle. Moreover the limit cycle exists if and only if $ad - bc > 0$ and $a(a\alpha + d) < 0$ and its stability is given by the sign of $-(a\alpha + d)$.

It follows by writing the system as a Liénard equation and then applying some classical results on these equations.

Theorem (part 1)

Consider random system

$$\dot{x} = Af(x) + By, \quad \dot{y} = Cf(x) + Dy,$$

where $f(x) = \alpha x^k + \sum_{k<i<m} f_i x^i + \beta x^m$, with $\alpha\beta \neq 0$, $k \leq m$ odd integers, $m > 1$, and A, B, C, D iid random variables with distribution $N(0,1)$. Assume also that $x = 0$ is the unique real root of $f(x) = 0$. Then

(i) When $k > 1$ the probability of having an odd number of limit cycles is $1/8$, and the probability of not having limit cycles or to have an even number is $7/8$.

(ii) When $k = 1$ and $\beta > 0$, the probability of having an odd number of limit cycles is $P^+(\alpha) \leq 1/2$, and the probability of not having limit cycles or to have an even number is $1 - P^+(\alpha)$.

(iii) When $k = 1$ and $\beta < 0$, the same results that in item (ii) hold but changing P^+ by P^-, where $P^-(\alpha) = P^+(-\alpha)$.

In all the cases, each limit cycle is counted with its own multiplicity.

Theorem (part 2)

The function $P^+ : \mathbb{R} \to (0, 1/2)$ is

$$P^+(\alpha) = \frac{1}{4\pi^2} \iiiint_{T(\alpha)} \mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}} \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}d,$$

where $T(\alpha) = \{(a, b, c, d) \,:\, ad - bc > 0,\ a(a\alpha + d) < 0\}$ is a decreasing function that satisfies

$$\lim_{\alpha \to -\infty} P^+(\alpha) = 1/2, \quad P^+(0) = 1/8, \quad \lim_{\alpha \to +\infty} P^+(\alpha) = 0,$$

The probabilities are obtained by studying the corresponding integrals

$$P^{+}(\alpha) = \frac{1}{4\pi^2} \iiiint_{T(\alpha)} \mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}} \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}d,$$

where the sets

$$T(\alpha) = \{(a, b, c, d) \, : \, ad - bc > 0, \, a(a\alpha + d) < 0\},$$

are defined by the conditions of existence of limit cycles fixed in previous results for the deterministic systems.

Corollary

For random system $\dot{x} = Ax^k + By, \quad \dot{y} = Cx^k + Dy$, $k > 1$ odd,

- the probability of having one limit cycle is $1/8$, and
- the probability of not having limit cycles is $7/8$.

This result is a consequence of previous theorem because for the deterministic system

$$\dot{x} = ax^k + by, \quad \dot{y} = cx^k + by,$$

the limit cycle (which is unique and hyperbolic) exists $\Leftrightarrow$ $ad - bc > 0$ and $ad < 0$. In this case it is not difficult to prove that

$$\frac{1}{4\pi^2} \iiiint_{\{(a,b,c,d)\,:\,ad-bc>0,\,ad<0\}} \mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d = \frac{1}{8},$$

Proposition

Consider random system

$$\dot{x} = A(\alpha x + x^3) + By, \quad \dot{y} = C(\alpha x + x^3) + Dy,$$

with $\alpha > 0$ and A, B, C, D iid random variables with distribution $N(0, 1)$. Then, for each $\varepsilon > 0$ there exists α big enough such that it has limit cycles with a positive probability, smaller that ε. Moreover, when the limit cycle exists it is unique.

Proposition

Consider random system

$$\dot{x} = A(\alpha x + x^3) + By, \quad \dot{y} = C(\alpha x + x^3) + Dy,$$

with $\alpha > 0$ and A, B, C, D iid random variables with distribution $N(0,1)$. Then, for each $\varepsilon > 0$ there exists α big enough such that it has limit cycles with a positive probability, smaller that ε.

we use the uniqueness of limit cycle for the associated deterministic system, that it exists if and only if $ad - bc > 0$, $a(a\alpha + d) < 0$, and that

$$\lim_{\alpha \to \infty} P^+(\alpha) = 0.$$

This is so, because the set $T(\alpha) = \{(a, b, c, d) \,:\, ad - bc > 0,\ a(a\alpha + d) < 0\}$, shrinks to the empty set when $\alpha \to \infty$.

Thank you very much for your attention

Let us prove (i). We define the new iid random variables $X = AD$ and $Y = BC$. Note that the density function of the new variables X and Y is an even function. Then the joint density function of (X, Y), $h(x, y)$, is symmetric respect to the origin, that is $h(x, y) = h(-x, -y)$.

From Theorem 2, the probability of having an odd number of limit cycles is

$$p = P(\beta(X - Y) > 0, \beta X < 0).$$

Notice $X - Y$ and $Y - X$ have the same distribution, and the same happens with X and $-X$. Thus, independently of the sign of β,

$$p = P(X - Y > 0, X < 0) = P(X - Y < 0, X > 0).$$

Finally,

$$P(X - Y > 0, X < 0) = \iint_{\{(x,y)\in\mathbb{R}^2\,:\,x-y>0,x<0\}} h(x, y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{8}.$$

Again by Theorem 2, the probability of having none or an even number of limit cycles is the probability of the complementary set, modulus a set of zero measure, that is $7/8$.