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ε-neighborhoods of orbits in unfoldings

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Parabolic and hyperbolic orbits on $\mathbb{R}_+$

- (attracting) *parabolic* germ
  $f(z) = x - ax^{k+1} + \ldots \in \text{Diff}(\mathbb{R}_+, 0)$, $a > 0$, $k \in \mathbb{N}$,

$$f^j(x_0) \sim j^{-1/k},\ j \to \infty$$

- (attracting) *hyperbolic* germ
  $f(x) = \lambda x + \ldots$, $0 < \lambda < 1$,

$$f^j(x_0) \sim \lambda^j,\ j \to \infty$$

Orbit of f with initial point $x_0 \in (\mathbb{R}_+, 0)$:

$$\mathcal{O}_f(x_0) := \{x_n := f^{\circ n}(x_0) :\ n \in \mathbb{N}_0\}.$$

Fractal data: ε-neighborhoods of orbits of parabolic and hyperbolic diffeomorphisms

- a parabolic orbit of *multiplicity* k:

$$\ell(\mathcal{O}^f(x_0)_\varepsilon) \sim (2/a)^{\frac{1}{k+1}} \frac{k+1}{k} \varepsilon^{\frac{1}{k+1}} + \ldots + c(\rho, a)\varepsilon(-\log\varepsilon) + \\ + o\left(\varepsilon(-\log\varepsilon)\right),\ \varepsilon \to 0,$$

$$\dim_B(\mathcal{O}^f(x_0)) = 1 - \frac{1}{k+1},\ \mathcal{M}(\mathcal{O}^f(x_0)) = (2/a)^{\frac{1}{k+1}} \frac{k+1}{k},$$

- a hyperbolic orbit:

$$\ell(\mathcal{O}^f(x_0)_\varepsilon) \sim a(\lambda) \cdot \varepsilon(-\log\varepsilon) + o(\varepsilon(-\log\varepsilon)),\ \varepsilon \to 0,$$
$$\dim_B(\mathcal{O}^f(x_0)) = 1 - 1 = 0,\ \mathcal{M}(\mathcal{O}^f(x_0)) = +\infty,$$

Continuous time length of ε-neighborhoods of orbits

- $\ell(\mathcal{O}^f(x_0)_\varepsilon) = T_\varepsilon + N_\varepsilon = 2\varepsilon \cdot n_\varepsilon + f^{n_\varepsilon}(x_0) + 2\varepsilon$ (Tricot)
- critical index $\varepsilon \mapsto n_\varepsilon,\ \varepsilon \approx 0$, a step-function:

$$f^{n_\varepsilon}(x_0) - f^{n_\varepsilon+1}(x_0) \leq 2\varepsilon, \quad f^{n_\varepsilon-1}(x_0) - f^{n_\varepsilon}(x_0) > 2\varepsilon.$$

$\Rightarrow$ [R2013,14] non-existence of the full power-log asymptotic expansion

$\Rightarrow$ appearance of oscillatory coefficients in the expansion ($G(\tau_\varepsilon)$, [MRR22])

Continuous time length of ε-neighborhoods of orbits

- *embedding of a germ in a flow* as time-1 map $\leftrightarrow$ existence of the *Fatou coordinate*

$$\Psi \circ f - \Psi = 1, \;\; f^t := \Psi^{-1}(t + \Psi)$$

f_{τ_ε} such that $f^{\tau_\varepsilon}(x_0) - f^{\tau_\varepsilon + 1}(x_0) = 2\varepsilon$

- the continuous-time length

$$\ell^c(\mathcal{O}^f(x_0)_\varepsilon) = 2\varepsilon \cdot \tau_\varepsilon + f^{\tau_\varepsilon}(x_0) + 2\varepsilon$$

- $n_\varepsilon = \lfloor \tau_\varepsilon \rfloor \Rightarrow \ell^c(\mathcal{O}^f(x_0)_\varepsilon)$ coincides with $\ell(\mathcal{O}^f(x_0)_\varepsilon)$ in finitely many first terms, but has a full expansion without oscillations

Saddle-node unfoldings

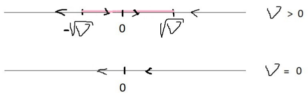
⋆ generic 1-parameter unfoldings of a non-hyperbolic singular point

$$\frac{dx}{dt} = F(x,\nu), \nu \geq 0, \qquad (1)$$

- F real analytic,
- singular point $x = 0$ at the bifurcation value $\nu = 0$ non-hyperbolic ($F(0,0) = 0$, $F_x(0,0) = 0$), generic assumptions:

$$F_\nu(0,0) \neq 0,\ \ F_{xx}(0,0) \neq 0. \qquad (2)$$

⇒ parabolic point at $x = 0$ bifurcates at $\nu = 0$ into two hyperbolic points: one attracting and one repelling, for $\nu > 0$



Weak formal model

$$\frac{dx}{dt} = F_{mod}(x,\nu),\ \ F_{mod}(x,\nu) := \frac{-x^2+\nu}{1+\rho(\nu)x},\ \ \nu \in [0,\delta). \qquad (3)$$

2 weak formal invariants:

- the value $\rho(0)$ is the parabolic *residual formal invariant*
- the multiplicity $k = 2$.

Weak formal equivalence [MardRoussRouss] – a formal change of variables

$$\hat{\Phi}(x,\nu) = (\hat{\varphi}_\nu(x), h(\nu)), \qquad (4)$$

h an analytic diffeomorphism such that $h(0) = 0$,
$\hat{\varphi}_\nu \in \mathbb{R}[[x]]$, $\nu \in [0,\delta)$, formal diffeo up to translation.

For simplicity, formal model (conclusions transported to non-model cases by formal conjugacy)

Formulation of the problem

- *uniform initial point:* $x_0 > 0$ small s.t. in attracting basin of 0 and of $\sqrt{\nu} > 0$, for small enough ν
- *discontinuity* in box dimension of orbits at the moment of bifurcation $\nu = 0$: hyperbolic 0 to parabolic $1 - \frac{1}{k+1}$
- $\ell^c(\mathcal{O}^{f_\nu}(x_0)_\varepsilon) \to \ell^c(\mathcal{O}^{f_0}(x_0)_\varepsilon)$, for every $\varepsilon \approx 0$, as $\nu \to 0$
- *but:* asymptotic terms in ε behave discontinuously, as $\nu \to 0$

$\Rightarrow$ Goal: find a Chebyshev scale so that the expansion, as $\varepsilon \to 0$, is uniform for all parameters $\nu \geq 0$

$\Rightarrow$ expansion in the so-called *Ecalle-Roussarie type compensators* – functions of ν, ε [Rouss98]

The uniform Fatou coordinate

By simple integration of (3):

$$\Psi_\nu(x) = \alpha(x-\sqrt{\nu}, 2\sqrt{\nu}) + \frac{\rho(\nu)}{2}\log(x^2-\nu),\ \nu \geq 0,$$

$\Rightarrow$ Expansion:

$$\Psi_\nu(x) \sim_{x\to\sqrt{\nu}} \begin{cases} \frac{1}{x}, & \nu = 0, \\ \left(\frac{\rho(\nu)}{2} - \frac{1}{2\sqrt{\nu}}\right)\log(x-\sqrt{\nu}), & \nu \neq 0. \end{cases}$$

The inverse Ecalle-Roussarie compensator

- ν and x small. The *Écalle-Roussarie compensator* [Rouss98]:

$$\omega(x,\nu) := \frac{x^{-\nu}-1}{\nu}$$

- pointwise, $\omega(x,\nu) \to -\log x$, as $\nu \to 0$
- convergence uniform in x, if we multiply by x^{δ}, $\delta > 0$.

Definition (The inverse compensator, [HMRZ23])

] For $x > 0$ and $\nu \in [0,\delta)$,

$$\alpha(x,\nu) := \frac{1}{\nu}\log\left(1+\frac{\nu}{x}\right).$$

the inverse compensator.

The inverse Ecalle-Roussarie compensator

- The name:

$$\alpha(x,\nu) = -\log\circ\,\omega^{-1}\left(\frac{1}{x},\nu\right),$$

ω^{-1} the inverse of ω wrt variable x;

- $\alpha(x,\nu)\to\frac{1}{x}$, pointwise as $\nu\to 0$;
- $x^{1+\delta}\alpha(x,\nu)\to x^{\delta}$, as $\nu\to 0$, *uniformly in* $x>0$;

- The asymptotic behavior, as $x\to 0$:

$$\alpha(x,\nu) = \begin{cases}\frac{1}{x}, & \nu=0,\\ \frac{1}{\nu}(-\log x)+\frac{\log\nu}{\nu}+\mathbb{R}_\nu[[x]], & \nu\neq 0.\end{cases} \quad (5)$$

The inverse displacement function g_ν^{-1}

Displacement function $g_\nu := \mathrm{id} - f_\nu \in \mathrm{Diff}(\mathbb{R},0)$:

$$g_\nu(x) = \left(1 - e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}\right) \cdot (x - \sqrt{\nu}) + \\ + e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}} \cdot a\Big(\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}\Big) \cdot \frac{1+\rho(\nu)\sqrt{\nu}}{(1-\rho(\nu)\sqrt{\nu})^2} \cdot (x-\sqrt{\nu})^2 + \\ + (x-\sqrt{\nu})^3 \cdot \mathbb{R}_\nu\{(x-\sqrt{\nu})\}, \quad \nu \in [0,\delta),\ x \to \sqrt{\nu}. \tag{6}$$

(linear for $\nu > 0$, quadratic for $\nu = 0$)

Square root type compensator

Definition (The square root compensator, HMR23)

ν, $x > 0$ *small,*

$$\tilde{\eta}(x,\nu) := \sqrt{x+\nu} - \sqrt{\nu},$$

the square root-type compensator.

- asymptotic expansion:

$$\tilde{\eta}(x,\nu) = \begin{cases} \sqrt{x}, & \nu = 0, \\ \frac{x}{\sqrt{\nu}} + \sqrt{\nu}\,\frac{x^2}{\nu^2}\,\mathbb{R}\{\frac{x}{\nu}\}, & \nu > 0,\ x \to 0. \end{cases} \tag{7}$$

- $\tilde{\eta}$ *small*, for small x,
- $\tilde{\eta}(x,\nu) \to \sqrt{x}$ uniformly in x, as $\nu \to 0+$.

The uniform Chebyshev expansion in ε, $\nu \geq 0$

- $$\ell^c(T_{\varepsilon,\nu}) = \big(\Psi_\nu((g_\nu)^{-1}(2\varepsilon)) - \Psi_\nu(x_0)\big) \cdot 2\varepsilon,$$
- g_ν quadratic for $\nu = 0$, linear for $\nu > 0$ $\Rightarrow g_\nu^{-1}(\varepsilon)$ contains a square-root type compensator
- bad: composition of two compensators

$\Rightarrow$ change the variable from ε to

$$\eta := \theta_{-\sqrt{\nu}} \circ g_\nu^{-1}(2\varepsilon)$$

$\Rightarrow$ variable η contains the parameter, same as E-R compensator variable $x\omega$ [Rouss98]

The uniform Chebyshev expansion in ε, $\nu \geq 0$

Theorem 1

In compensator variable $\eta \geq 0$, the continuous tail $\ell^c(T_{\varepsilon,\nu})$ admits an asymptotic expansion, uniform in parameter $\nu \in [0, \delta)$, in the following Chebyshev scale:

$$\left\{ I(\nu,\eta)\eta,\ I(\nu,\eta)\eta^2,\ I(\nu,\eta)\eta^3, \ldots \right\},$$

as $\eta \to 0$, where

$$I(\nu,\eta) := \alpha(\eta, 2\sqrt{\nu}) + \frac{\rho(\nu)}{2} \log\left(\eta^2 + 2\sqrt{\nu}\cdot\eta\right) - \Psi_\nu(x_0), \quad (8)$$

non-zero in the uniform neighborhood $\eta \in [0, d)$ for $\nu \in [0, \delta)$.

The uniform Chebyshev expansion in ε, $\nu \geq 0$; reading the codimension of bifurcation

Theorem 2

More precisely,

$$\ell(T^c_{\varepsilon,\nu}) = I(\nu,\eta) g_\nu(\eta+\sqrt{\nu})$$

$$\sim \left(1 - e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}\right) \cdot I(\nu,\eta)\eta + \quad (9)$$

$$+ e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}} a\left(\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}\right) \frac{1+\rho(\nu)\sqrt{\nu}}{(1-\rho(\nu)\sqrt{\nu})^2} \cdot I(\nu,\eta)\eta^2 + \quad (10)$$

$$+ o_\nu(I(\nu,\eta)\eta^2),\ \eta \to 0+ .$$

- at $\nu = 0$, the first non-zero coefficient is the third one.
- *Rolle in Chebyshev scales:* at most two zero points bifurcate in $\eta \mapsto \ell^c(T_{\varepsilon,\nu})$ from the zero point $\eta = 0$ at $\nu = 0$ (codimension of bifurcation 2)

Expansion by *indecomposable* compensators $\alpha, \tilde{\eta}$

Relation of η to the simpler (indecomposable) compensator $\tilde{\eta}$:

$$\eta(2\varepsilon, \nu) = \chi_\nu\Big(\tilde{\eta}\big(\frac{2\varepsilon}{C(0)}, r(\nu)\big)\Big), \ \nu \in [0, \delta),$$

$r(\nu) := \frac{1-e^{-\frac{2\sqrt{\nu}}{1-\rho(\nu)\sqrt{\nu}}}}{2C(0)}$,
χ_ν is a germ of a diffeo tangent to id.

- Another compensator variable:

$$\kappa(x, \nu) := \frac{1}{x+\nu}. \tag{11}$$

- $$\frac{d}{dx}\alpha(x, \nu) = -\frac{1}{x}\kappa(x, \nu).$$
- $\kappa(x, \nu) \to \frac{1}{x}$ by points, as $\nu \to 0$
- $x^\delta \kappa(x, \nu) \to x^{-1+\delta}$ uniformly as $\nu \to 0$, $\delta > 0$.

Re-arrangement of the terms in the expansion:

1. the terms that give the same power-logarithmic asymptotic term in $\tilde{\eta}$ for $\nu = 0$ grouped together as a block inside square brackets;

2. $\nu > 0$: each block possibly *infinite* – can be further expanded asymptotically in a convergent power-logarithmic series in $\tilde{\eta}$, as $\tilde{\eta} \to 0$.

Expansion by *indecomposable* compensators α, $\tilde{\eta}$

$$\ell^c(T_{\varepsilon,\nu}) \sim \left(1 - e^{-\frac{2\sqrt{h(\nu)}}{1-\rho(h(\nu))\sqrt{h(\nu)}}}\right) \cdot \Bigg\{[\alpha(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}) \cdot \tilde{\eta}] +$$

$$+ \frac{\rho(h(\nu))}{2} \sum_{k=0}^{\infty} a_k(\nu) \Big[\log \tilde{\eta} \cdot \tilde{\eta}^{k+1} + \log \big(\tilde{\eta} + \frac{2\sqrt{h(\nu)}}{C(\nu)}\big) \cdot \tilde{\eta}^{k+1} \Big] +$$

$$+ \sum_{k=1}^{\infty} \Big[a_k(\nu) \cdot \alpha\big(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}\big) \cdot \tilde{\eta}^{k+1} + N_k^{\nu}\big(\tilde{\eta}, \kappa\big(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}\big)\big) \Big] \Bigg\} +$$

$$+ c_2(\nu) \cdot \Bigg\{ [\alpha(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}) \cdot \tilde{\eta}^2] +$$

$$+ \frac{\rho(h(\nu))}{2} \sum_{k=0}^{\infty} b_k(\nu) \Big[\log \tilde{\eta} \cdot \tilde{\eta}^{k+2} + \log \big(\tilde{\eta} + \frac{2\sqrt{h(\nu)}}{C(\nu)}\big) \cdot \tilde{\eta}^{k+2} \Big] +$$

$$+ \sum_{k=1}^{\infty} \Big[b_k(\nu) \cdot \alpha\big(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}\big) \cdot \tilde{\eta}^{k+2} + M_{k+1}^{\nu}\big(\tilde{\eta}, \kappa\big(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}\big)\big) \Big] \Bigg\},$$

$c_2(0) \neq 0$, N_k^{ν}, M_k^{ν} homogenous polynomials of degree k whose coefficients depend on ν.

Expansions in separate cases $\nu = 0$ and $\nu > 0$

- $\nu = 0$, $\tilde{\eta} = \sqrt{\frac{2\varepsilon}{C(0)}}$,

$$\ell^c(T_{\varepsilon,0}) \sim \frac{c_2(0)}{C(0)}\tilde{\eta} + \rho(0)\sum_{k=0}^{\infty} b_k(0)\tilde{\eta}^{k+2}\log\tilde{\eta} + \sum_{k=1}^{\infty} c_k\tilde{\eta}^{k+2}$$

$$= \frac{c_2(0)\sqrt{2}}{C(0)^{3/2}}\varepsilon^{\frac{1}{2}} + \rho(0)\sum_{k=0}^{\infty} c_k\varepsilon^{\frac{k+2}{2}}\log\varepsilon + \sum_{k=1}^{\infty} d_k\varepsilon^{\frac{k+2}{2}},\ \ \varepsilon \to 0,$$

- $\nu > 0$,
  - $$\tilde{\eta} = \frac{2\varepsilon}{C(0)\sqrt{r(h(\nu))}} + o(\varepsilon) \in \mathbb{R}_\nu\{\varepsilon\}.$$
  - $$\alpha(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}) \sim \log\varepsilon + \frac{1}{2\sqrt{h(\nu)}}\log\frac{C(0)\sqrt{r(h(\nu))h(\nu)}}{C(\nu)} + \varepsilon\mathbb{R}_\nu\{\varepsilon\}$$
  - $$\log\left(\tilde{\eta} + \frac{2\sqrt{h(\nu)}}{C(\nu)}\right) \sim \log\frac{2\sqrt{h(\nu)}}{C(\nu)} + \frac{C(\nu)}{C(0)\sqrt{r(h(\nu))h(\nu)}}\varepsilon + \varepsilon^2\mathbb{R}_\nu\{\varepsilon\}$$
  - $$\kappa(C(\nu)\tilde{\eta}, 2\sqrt{h(\nu)}) \sim \frac{1}{2\sqrt{h(\nu)}} - \frac{C(\nu)}{2C(0)h(\nu)\sqrt{r(h(\nu))}}\varepsilon + \varepsilon^2\mathbb{R}_\nu\{\varepsilon\},\ \varepsilon \to 0.$$

$\rightarrow$ monomials in the expansion are ε^k, $k \in \mathbb{N}_0$, and $\varepsilon^k \log \varepsilon$, $k \geq 1$.