Period function of planar turning points

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Bifurcations of Dynamical Systems and Numerics
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Joint work with Renato Huzak

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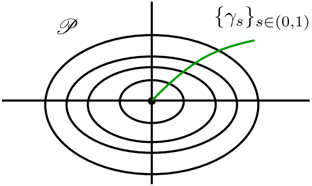
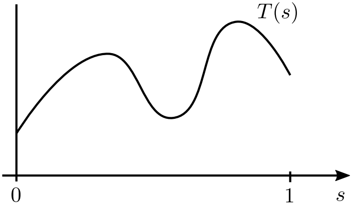
The period function





Critical periods

Isolated zeros of $T'(s)$.

Liénard center

Let $f(x)$ be a polynomial of degree $n = 2\ell - 1$, the Liénard equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - yf(x), \end{cases}$$

has a center if and only if f is an odd polynomial.

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Writing $F(x) = \int_0^x f(s)ds$ and replacing y by $y - F(x)$,

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -x. \end{cases}$$

Where $F(x)$ is an even polynomial of degree $n + 1 = 2\ell$ with $F(0) = 0$.

Liénard center

P. De Maesschalck, F. Dumortier. The period function of classical Liénard equations. *J. Differential Equations* 233 (2007) 380–403.

Theorem

For each choice of an odd integer n, there exists a polynomial F of degree $n+1=2\ell$ so that the Liénard system has at least $n-1=2\ell-2$ critical periods.

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Conjecture

For any odd n an upperbound for the number of critical periods that a classical Liénard system of degree $n+1$ can have is given by $n-1$.

Compactification

Any Liénard system of degree exactly 2ℓ is linearly equivalent to some

$$S_{\epsilon,a} : \begin{cases} \dot{x} = y - \left(x^{2\ell} + \sum_{k=1}^{\ell-1} a_{2k} x^{2k} \right), \\ \dot{y} = -\epsilon x, \end{cases}$$

or to some

$$L_{\lambda} : \begin{cases} \dot{x} = y - \left(x^{2\ell} + \sum_{k=1}^{\ell-1} \lambda_{2k} x^{2k} \right), \\ \dot{y} = -x. \end{cases}$$

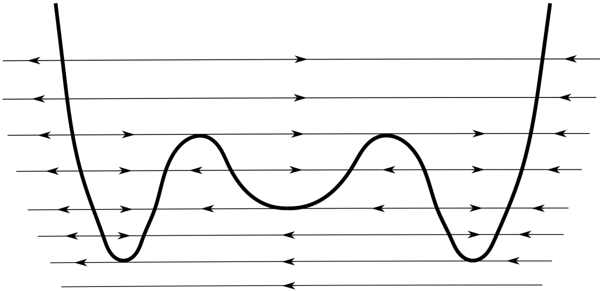
$a = (a_2, a_4, \dots, a_{2\ell-2}) \in \mathbb{S}^{\ell-2}$, $\epsilon \in [0, \epsilon_0]$,
$\lambda = (\lambda_2, \lambda_4, \dots, \lambda_{2\ell-2}) \in B(0, K)$.

F. Dumortier. Compactification and desingularization of spaces of polynomial Liénard equations. *J. Differential Equations* 224 (2006) 296–313.

Compactification

Slow-fast Liénard system

$$S_{0,a}:\begin{cases}\dot{x}=y-\left(x^{2\ell}+\sum_{k=1}^{\ell-1}a_{2k}x^{2k}\right),\\ \dot{y}=0.\end{cases}$$



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Slow-fast Liénard system

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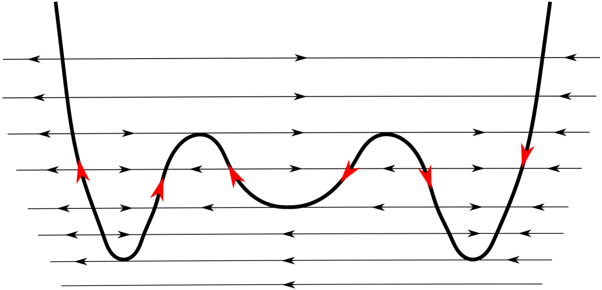
$$y = F(x) \Rightarrow \dot{y} = F'(x)\dot{x} = -\epsilon x \Rightarrow \frac{\dot{x}}{\epsilon} = \frac{-x}{F'(x)} \Rightarrow x' = \frac{-x}{F'(x)}$$



Periodic orbits in slow-fast Liénard $S_{\epsilon,a}$ are perturbations of **slow-fast limit periodic sets**.

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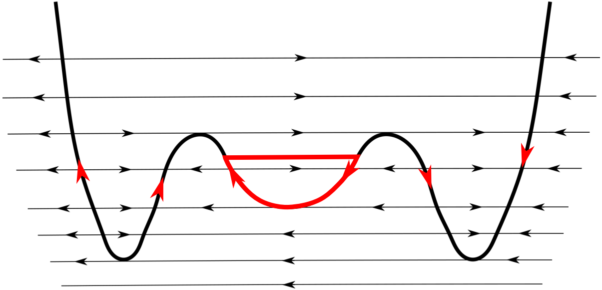

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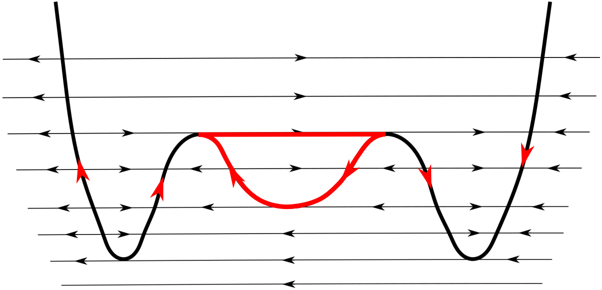

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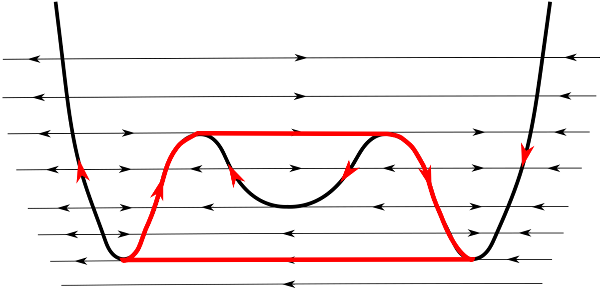

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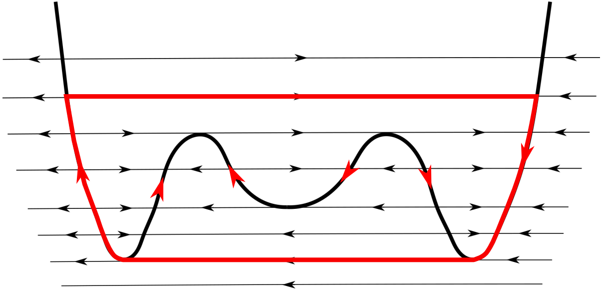

Periodic orbits in slow-fast Liénard $S_{\epsilon,a}$ are perturbations of **slow-fast limit periodic sets**.

Time analysis in slow-fast Liénard systems

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Idea

For $\epsilon \approx 0$ the period function increase if longer distance is travelled near the critical curve $y=F(x)$.
They choose $F(x)$ to be the Legendre polynomial of degree 2ℓ,

$$F(x)=\frac{1}{4^{\ell}(2\ell)!}\frac{d^{2\ell}}{dx^{2\ell}}((x^2-1)^{2\ell}),$$

has $2\ell-1$ critical points with increasing critical levels.

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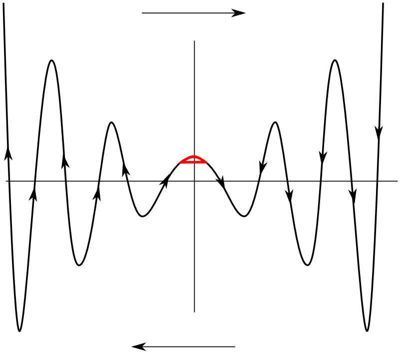

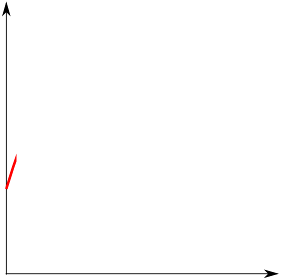

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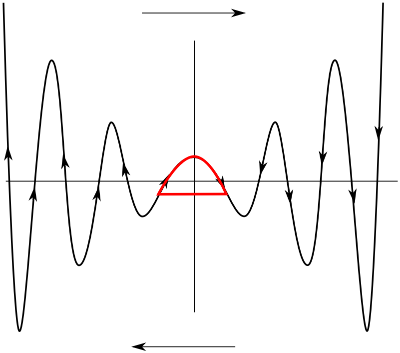

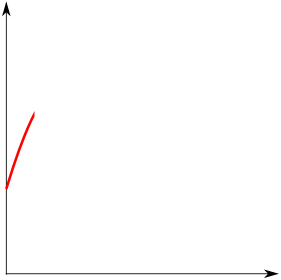

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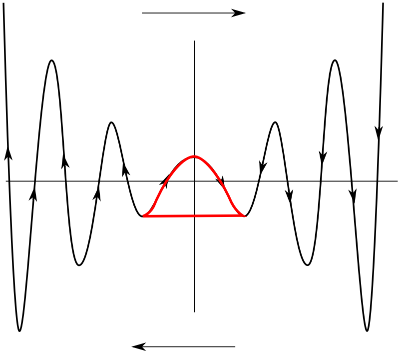

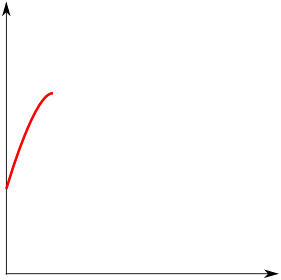

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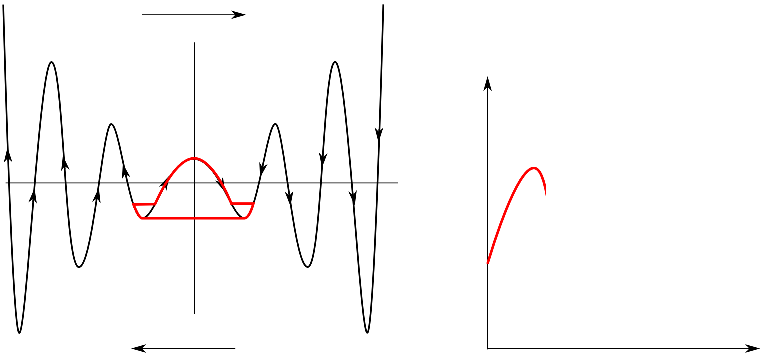

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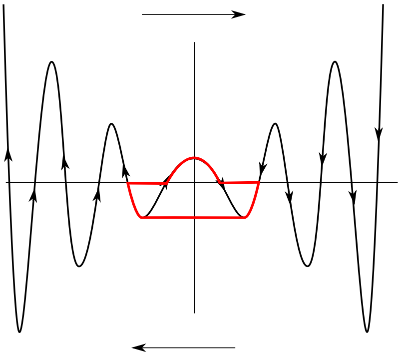

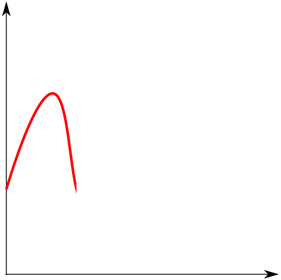

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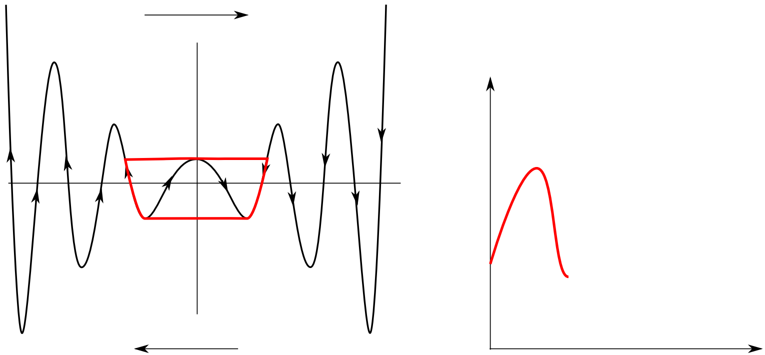

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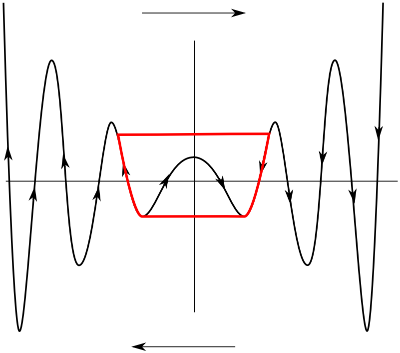

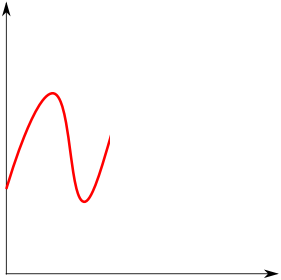

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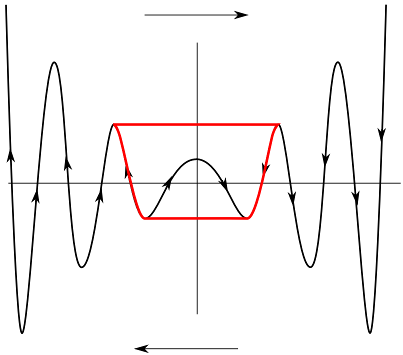

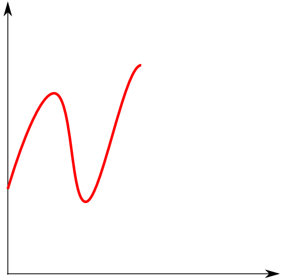

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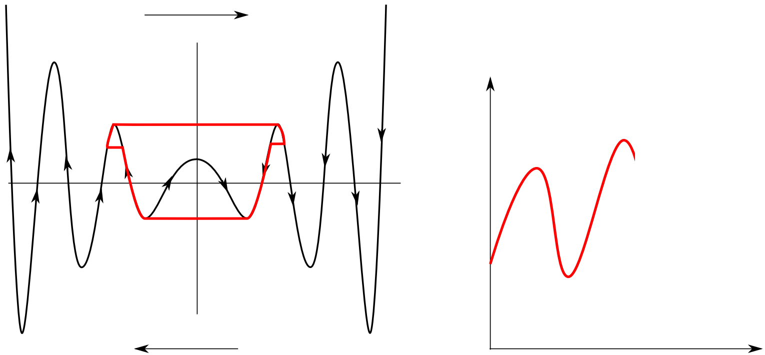

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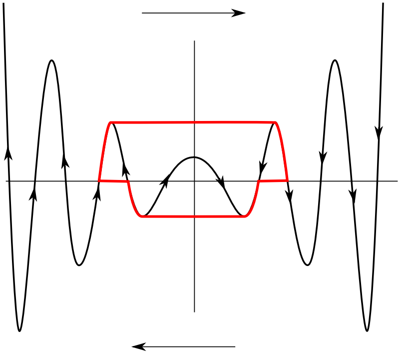

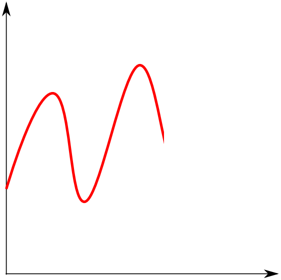

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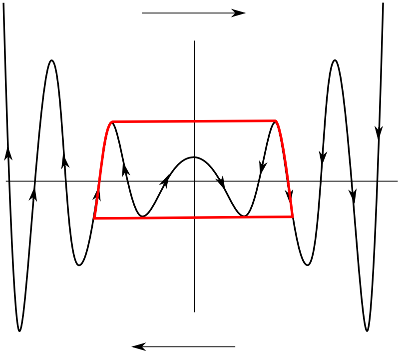

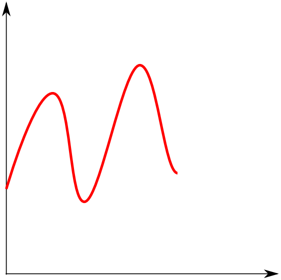

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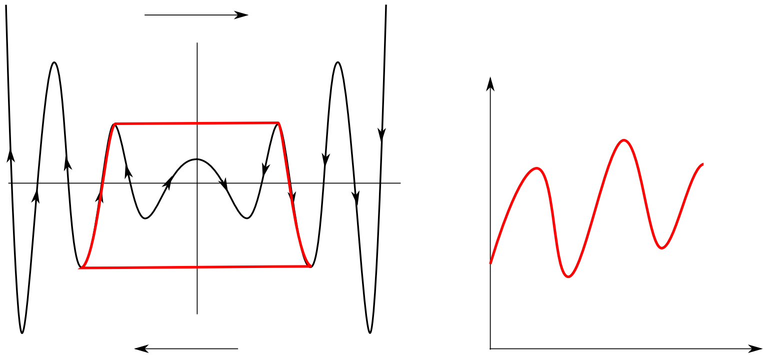

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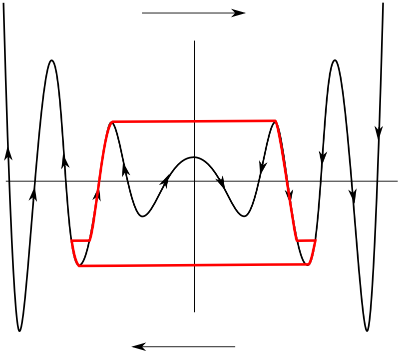

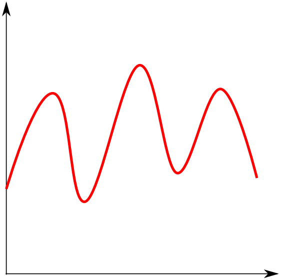

Period function at turning points

Conjecture

For any odd n an upperbound for the number of critical periods that a classical Liénard system of degree $n+1$ can have is given by $n-1$.

Our contribution

Study the period function of planar generic and non-generic turning points (the origin for slow-fast Liénard systems).

Period function at turning points

We consider slow-fast polynomial Liénard equations of center type

$$X_{\epsilon,\eta}: \begin{cases} \dot{x} = y - (x^{2n} + \sum_{k=1}^{\ell} a_k x^{2n+2k}), \\ \dot{y} = \epsilon^{2n}(-x^{2n-1} + \sum_{k=1}^{m} b_k x^{2n+2k-1}). \end{cases}$$

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The Theorem

Let $\ell, m \geq 1$ and $n = 1$ (resp. $n > 1$) be fixed. For any compact $K \subset \mathbb{R}^{\ell+m}$ there exist $\epsilon_0 > 0$ and $y_0 > 0$ small enough such that the period function $T(y;\epsilon)$ of the center of $X_{\epsilon,\eta}$ at the origin is strinctly monotonous increasing (resp. has a global minimum) in the interval $]0, y_0]$ for all $\epsilon \in]0, \epsilon_0]$ and $\eta \in K$.

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Corollary

When $n = 1$ (generic case) and $b_k = 0$ we recover classic slow-fast Liénard with $F(x) = a_2 x^2 + \cdots$, $a_2 > 0$.

Period function at turning points

The family blow-up

Desingularize the system $X_{\epsilon,\eta}$ near $(x, y, \epsilon) = (0, 0, 0)$.

$$\Psi : \mathbb{R}^+ \times \mathbb{S}^2_+ \to \mathbb{R}^3 : (r, (\bar{x}, \bar{y}, \bar{\epsilon})) \mapsto (x, y, \epsilon) = (r\bar{x}, r^{2n}\bar{y}, r\bar{\epsilon}), \bar{\epsilon} \geq 0.$$

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The blown-up vector field is the pullback

$$\bar{X}_\eta := \frac{1}{r^{2n} - 1}\Psi^* \left(X_{\epsilon,\eta} + 0\frac{\partial}{\partial \epsilon} \right).$$

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To study the blown-up v.f. $\bar{X}_\eta$ near the blow-up locus $\{0\} \times \mathbb{S}^2_+$ we use different charts with rectified coordinates.

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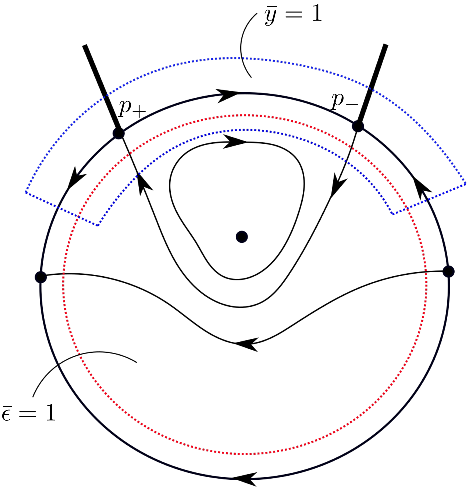

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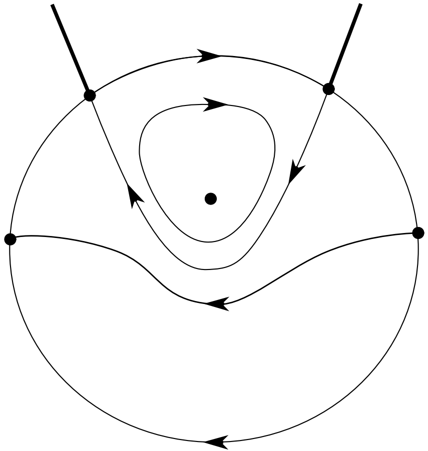

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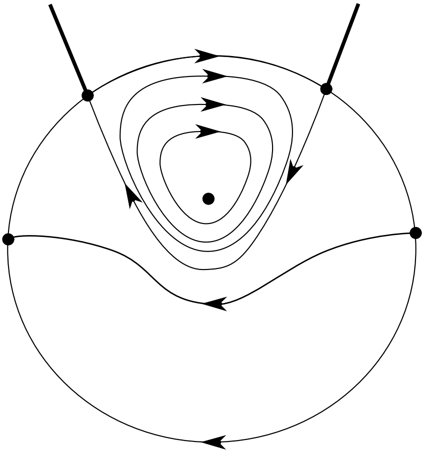

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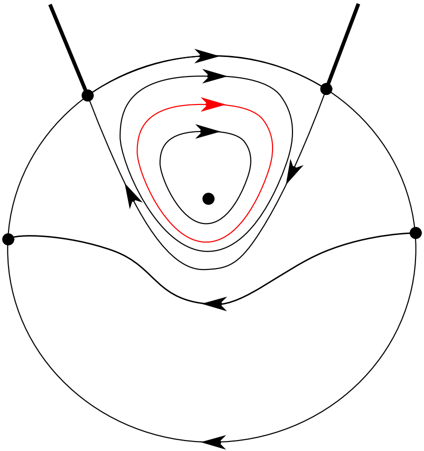

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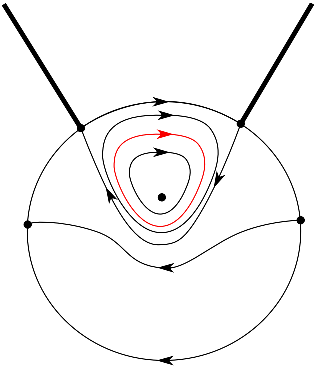

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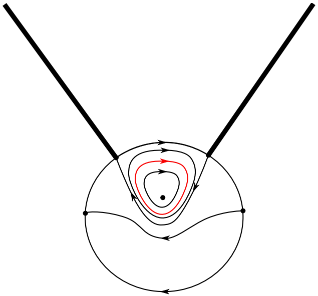

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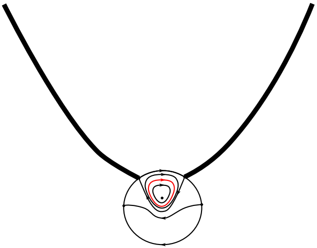

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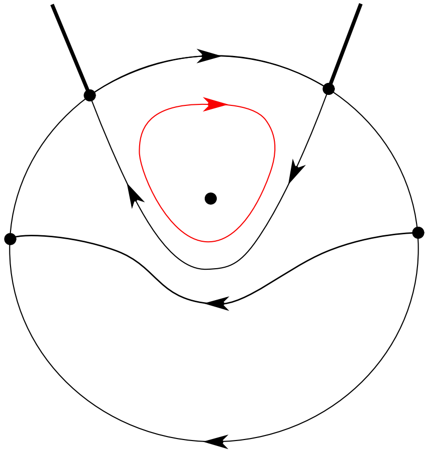

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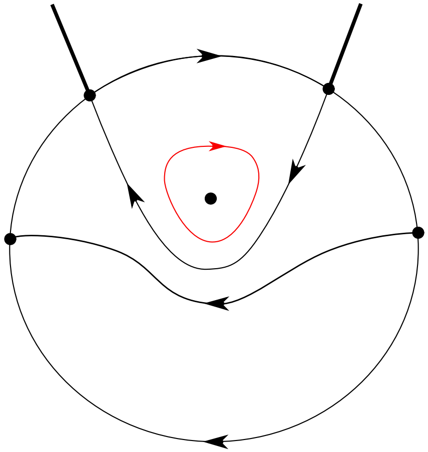

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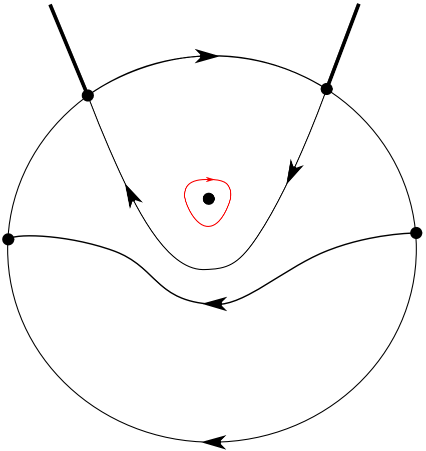

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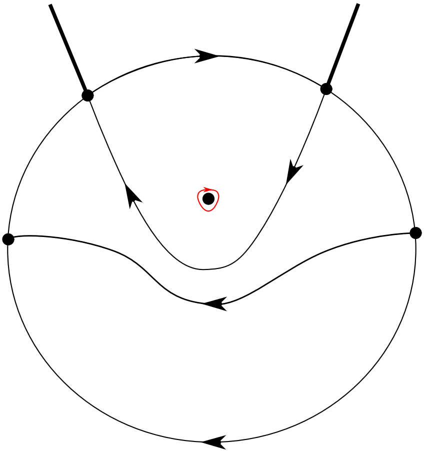

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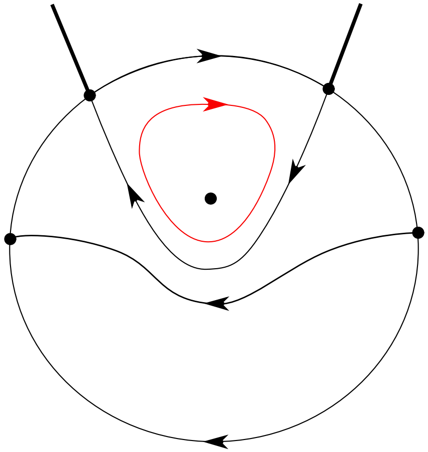

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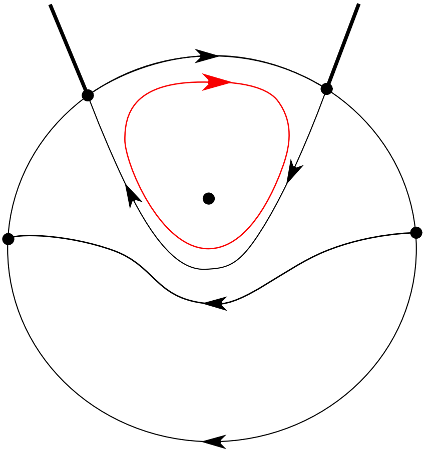

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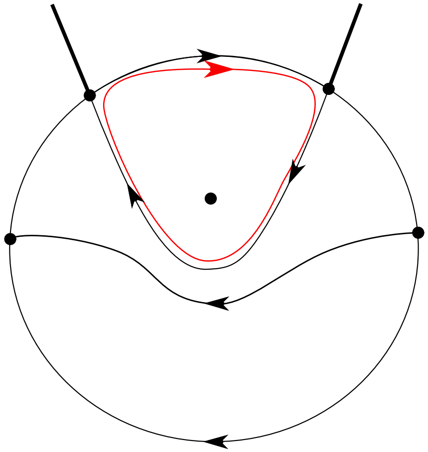

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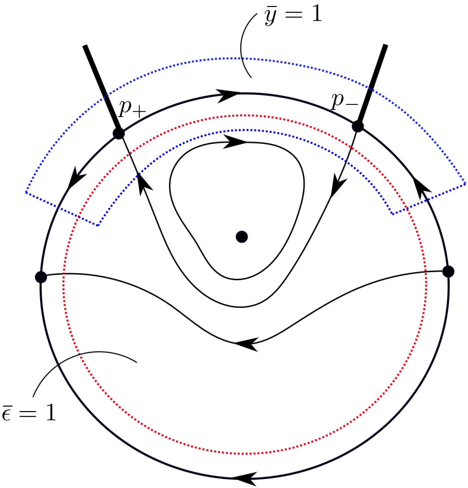

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The family blow-up

In $\{\bar{\epsilon} = 1\}$ we have $(x, y, \epsilon) = (r\bar{x}, r^{2n}\bar{y}, r)$ and $X_{\epsilon,\eta}$ becomes $X_F := \epsilon^{2n-1}\bar{X}_F$,

$$\bar{X}_F : \begin{cases} \dot{\bar{x}} = \bar{y} - \left(\bar{x}^{2n} + \sum_{k=1}^{\ell} a_k \epsilon^{2k} \bar{x}^{2n+2k}\right) \\ \dot{\bar{y}} = -\bar{x}^{2n-1} + \sum_{k=1}^{m} b_k \epsilon^{2k} \bar{x}^{2n+2k-1}. \end{cases}$$

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When $\epsilon = 0$,

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \bar{x}^{2n} \\ \dot{\bar{y}} = -\bar{x}^{2n-1}, \end{cases}$$

with first integral $H(\bar{x}, \bar{y}) = e^{-2n\bar{y}}(\bar{y} - \bar{x}^{2n} + 1/2n)$.

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Idea

We use classical tools for study the period function near the origin $(\bar{x}, \bar{y}) = (0, 0)$ and on the period annulus.

Period function at turning points

The family blow-up: bifurcation at the origin

For $\epsilon \approx 0$ small,

$$T_F(\bar{x}, \epsilon) = T_0(\bar{x}) + O(\epsilon)$$

where $T_0(\bar{x})$ is the period function of

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \bar{x}^{2n} \\ \dot{\bar{y}} = -\bar{x}^{2n-1}, \end{cases}$$

If $n = 1$ the center is quadratic. Chicone and Jacobs:

$$T_0(\bar{x}) = 2\pi + \frac{\pi}{3}\bar{x}^2 + O(\bar{x}^3)$$

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If $n > 1$ a little more technical but $T_0(\bar{x}) \to +\infty$ and $\frac{d}{d\bar{x}} T_0(\bar{x}) \to -\infty$ as $\bar{x} \to 0^+$.

Period function at turning points

The family blow-up: bifurcation at the interior

The change of variables $u = \ln(1 + 2n(\bar{y} - \bar{x}^{2n}))$, $v = \bar{x}$ transforms the system with $\epsilon = 0$ into the Hamiltonian with separate variables

$$\begin{cases} \dot{u} = -2nv^{2n-1}, \\ \dot{v} = V_n'(u) \end{cases}$$

with $V_n(u) = \frac{1}{2n}(e^u - u - 1)$.

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with $V_n(u) = \frac{1}{2n}(e^u - u - 1)$.
If $n = 1$ we use Schaaf criterion of monotonicity for potential systems.
If $n > 1$ we use a criterion of strict convexity due to Sabatini for Hamiltonians of the form $H(u, v) = G(u) + F(v)$ with $G(u) = \alpha u^{2k} + o(u^{2k})$ and $F(v) = \beta v^{2\ell} + o(v^{2\ell})$.

Period function at turning points

The family blow-up: bifurcation at the polycycle

In $\{\bar{y} = 1\}$ we have $(x, y, \epsilon) = (RX, R^{2n}, RE)$ and $X_{\epsilon,\eta}$ becomes $X_D := R^{2n-1}\bar{X}_D$,

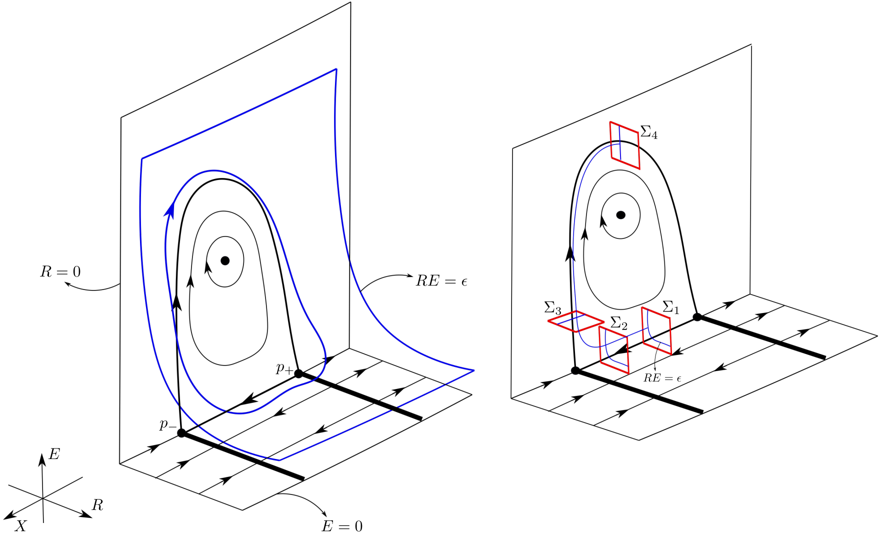
$$\bar{X}_D : \begin{cases} \dot{X} = 1 - (X^{2n} + \sum_{k=1}^{\ell} a_k R^{2k} X^{2n+2k}) + \frac{1}{2n} X E^{2n} G(X, R, \eta), \\ \dot{R} = -\frac{1}{2n} R E^{2n} G(X, R, \eta), \\ \dot{E} = \frac{1}{2n} E^{2n+1} G(X, R, \eta), \end{cases}$$

with $G(X, R, \eta) = X^{2n-1} - \sum_{k=1}^{m} b_k R^{2k} X^{2n+2k-1}$.

For $R = E = 0$ the system has semi-hyperbolic singularities at $X = -1$ (p_+) and $X = 1$ (p_-).

Period function at turning points

The family blow-up: bifurcation at the polycycle



Period function at turning points

The family blow-up: bifurcation at the polycycle

$(R_1, E_1) \in \Sigma_1$, $(R_2, E_2) \in \Sigma_2$, $(\bar{x}, \epsilon) \in \Sigma_3$.

$$T(R_1, E_1) = T_{1,2}(R_1, E_1) + T_{2,3}(R_2, E_2) + T_{3,4}(\bar{x}, \epsilon)$$

$$T_{1,2}(R_1, E_1) = \frac{1}{R_1^{2n-1}} I(R_1, E_1), \;\; I > 0 \;\; C^\infty$$

$$T_{2,3}(R_2, E_2) = \frac{2n}{(R_2 E_2)^{2n-1}} \int_{E_2}^{E_3} \frac{dE}{E^2 \kappa(E, R_2, E_2)}, \;\; \kappa > 0 \text{ bounded}$$

$$T_{3,4}(\bar{x}, \epsilon) = \frac{1}{\epsilon^{2n-1}} \bar{I}(\bar{x}, \epsilon), \;\; \bar{I} \;\; C^\infty$$

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$R_1 E_1 = R_2 E_2 = \epsilon$

Period function at turning points

The family blow-up: bifurcation at the polycycle

$$T(R_1, E_1) = \frac{2n}{(R_1 E_1)^{2n}} \left(\int_{E_2}^{E_3} \frac{dE}{E^2 \kappa(E, R_2, E_2)} + I(R_1, E_1) \right),$$

I bounded C^∞.
We take the Lie derivative $\mathcal{L}$

$$\mathcal{L}T := R_1 \frac{\partial T}{\partial R_1} - E_1 \frac{\partial T}{\partial E_1}$$

and we show, after computations, $\mathcal{L}T > 0$ and tending to infinity as $\epsilon \to 0$.

Gluing together

Let $n \geq 1$ and $T(y; \epsilon)$ the period function of the center at the origin of $X_{\epsilon,\eta}$. For each $\epsilon > 0$, $\epsilon \approx 0$ we consider the intervals $]0, \epsilon^{2n}\bar{y}_0]$, $[\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $[\epsilon^{2n}\bar{y}_3, y_0]$, where $\bar{y}_0, \bar{y}_1, y_0 > 0$ are small and independent of ϵ, and $\bar{y}_2, \bar{y}_3 > 0$ are large and independent of ϵ.

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- In $]0, \epsilon^{2n}\bar{y}_0]$ we use the local result in $\{\bar{\epsilon} = 1\}$ and there are no critical periods.

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- In $]0, \epsilon^{2n}\bar{y}_0]$ we use the local result in $\{\bar{\epsilon} = 1\}$ and there are no critical periods.
- In $[\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ we use the global result in $\{\bar{\epsilon} = 1\}$ showing that if $n = 1$ the period is increasing and if $n > 1$ there is a global minimum.

Gluing together

Let $n \geq 1$ and $T(y;\epsilon)$ the period function of the center at the origin of $X_{\epsilon,\eta}$. For each $\epsilon > 0$, $\epsilon \approx 0$ we consider the intervals $]0, \epsilon^{2n}\bar{y}_0]$, $[\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $[\epsilon^{2n}\bar{y}_3, y_0]$, where $\bar{y}_0, \bar{y}_1, y_0 > 0$ are small and independent of ϵ, and $\bar{y}_2, \bar{y}_3 > 0$ are large and independent of ϵ. For $\bar{y}_0, y_0$ small and $\bar{y}_3$ large, it suffices to decrease $\bar{y}_1$ and increase $\bar{y}_2$ to cover $]0, y_0]$.

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- In $[\epsilon^{2n}\bar{y}_3, y_0]$ we use the results in $\{\bar{y} = 1\}$ and there are no critical periods.

Many thanks for your attention

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