Computation of Normal Forms of ODEs for Systems with Many Parameters

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Outlines:

- Basics of normal form theory
- A module of formal vector fields with a special grading
- An algorithm for computing normal forms in the module
- Normal forms of generalized vector fields

Main questions in the normal form theory:

- analyze the structure of normal form,
- compute it,
- analyze convergence of the obtained series.

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• The classical approach developed by Poincare, Dulac and Lyapunov and requires substitution series into series.
The other one is based on Lie transformation and was developed starting from the works of Birkhoff, Steinberg, Chen and others. Using this way a normalization is performed as a sequence of linear transformations.

Notations:

- $\mathbb{F}$ a field ($\mathbb{R}$ or $\mathbb{C}$),
- $\mathcal{V}^n$ the space of vector fields $v : \mathbb{F}^n \to \mathbb{F}^n$ which coordinates are power series in $x_1, \ldots, x_n$ (we will say "in x") vanishing at the origin, which can be convergent or merely formal,
- $\mathcal{V}_j^n$ be the vector space of polynomial vector fields on $\mathbb{F}^n$, that is, vector fields $v : \mathbb{F}^n \to \mathbb{F}^n$ such that each component v_i $(i = 1, \ldots, n)$ of v is a homogeneous polynomial of degree j.

$\mathcal{V}^n$ and $\mathcal{V}_j^n$ are vector spaces over $\mathbb{F}$ and $\mathcal{V}^n = \oplus_{j=0}^{\infty} \mathcal{V}_j^n$.

$$\dot{x} = u(a, x), \tag{1}$$

$a = (a_1, \dots, a_\ell)$ is an ℓ-tuple of parameters, $u(a, x) \in \mathcal{V}^n$ and all terms of $u(a, x)$ depend polynomially on parameters a.
$\mathcal{V}^n$ can be viewed also as a module over the ring of power series in $x_1, \dots, x_n$.
It has natural grading by the degree of polynomials:

$$\dot{x} = \sum_{j=1}^{\infty} u_j(a, x), \tag{2}$$

where $u_j(a, x)$ is a homogeneous polynomial of degree j in x, that is, $u_j(a, x) \in \mathcal{V}_j^n$.
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• We use another approach which is based on the grading of the power series by the degree of polynomials in the parameters of the system:

$$\dot{x} = \sum_{s=1}^{\infty} \bar{u}_s(a, x), \tag{3}$$

where $\bar{u}_s$ is a homogeneous polynomial of degree s in variables a.
We work with formal vector fields using the grading of the formal series by polynomials which are *homogeneous with respect to parameters a* of $u(a, x)$.

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  Namely, for (1) the terms of powers series in a normal form have the form $p(a)x^r$, where x^r is a resonant monomial and $p(a) = \sum_{\alpha \in Supp(p)} p_\alpha a^\alpha$ is a polynomial in a. Using traditional methods one can compute only the whole term $p(a)x^r$.

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  We can compute any term $p_\alpha a^\alpha$ of $p(a)$ without computing the whole polynomial $p(a)$.
- Only arithmetic operations with numbers are used.

$$\dot{x} = f(x), \quad f(x) = Ax + \sum_{j=2}^{\infty} f_j(x), \qquad x \in \mathbb{F}^n, \tag{4}$$

$f_j(x)$ is an n-dimensional vector valued homogeneous polynomial of degree j. A is a *diagonal matrix* with the eigenvalues $\lambda_1, \ldots, \lambda_n$, $x = (x_1, \ldots, x_n)^T$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{F}^n$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ we denote

$$\langle \lambda, \alpha \rangle = \sum_{i=1}^{n} \alpha_i \lambda_i, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. x *is a column vector* and x^α *is a monomial.*
Normalizing transformation:

$$x = y + \tilde{h}(y) = y + \sum_{j=2}^{\infty} \tilde{h}_j(y). \tag{5}$$

$$\dot{y} = Ay + \tilde{g}(y), \quad \tilde{g}(y) = \sum_{j=2}^{\infty} \tilde{g}_j(y), \qquad \tilde{g}_j(y) \in \mathcal{V}_j^n, \qquad j = 2, 3, \ldots. \tag{6}$$

x^α in the k-th entry is a *resonant* monomial if if is of the form $g^{(\alpha)} y^\alpha e_k$ with

$$\langle \lambda, \alpha \rangle - \lambda_k = 0. \tag{7}$$

We say that system (6) is in the *Poincaré–Dulac normal form* (or, simply in the normal form) if $\tilde{g}(y)$ contains only resonant terms.

The *homological operator*:

$$(\pounds v)(x) = Dv(x)Ax - Av(x). \tag{8}$$

Let

$$\pounds_j : \mathcal{V}_j^n \to \mathcal{V}_j^n$$

be the restriction of $\pounds$ to $\mathcal{V}_j^n$. The $\pounds_j$ is semisimple with the eigenvalues $\beta_{\alpha,i} = \langle \alpha, \lambda \rangle - \lambda_i$ and (basis) eigenvectors $p_{i,\alpha} = e_i x^\alpha$ ($|\alpha| = j$, $i = 1, \ldots, n$). The kernel of $\pounds_j$ is spanned by all monomials $e_i x^\alpha$, where i and α satisfy $\beta_{\alpha,i} = 0$.

$$\mathcal{V}_j^n = \mathrm{im}\pounds_j \oplus \ker \pounds_j.$$

Assume that equation (4) has already been normalized to order $j-1$, so (6) is a normal form of (4) up to order $j-1$.

- Solve
$$\pounds_j \tilde{h}_j = f_j - \tilde{g}_j, \tag{9}$$
- With this $\tilde{g}_j$ (6) is in the normal form to order j.
- Perform the transformation $x \to x + \tilde{h}_j(x)$
- Re-expand the series $f(x)$ up to order $j+1$.

Another setting for computation the normal form is using the Lie brackets.

$$[w, v] = D(v)w - D(w)v$$

. The *adjoint map*: $w(x) \in \mathcal{V}^n$,

$$\operatorname{ad} w(x) = [w(x), \cdot] \, : \, \mathcal{V}^n \to \mathcal{V}^n.$$

In particular, $\pounds = \operatorname{ad} Ax$ and (6) is in the normal form iff

$$(\operatorname{ad} Ax)\, \tilde{g}_k = 0, \text{ for all } k = 2, 3, \dots.$$

Let $\dot{x} = a(x)$, $dx/ds = b(x)$, $b(0) = 0$ with the flow ψ_s. By the Theorem on Lie series for vector fields

$$\psi_s'(x)^{-1} a(\psi_s(x)) = a(x) + s(\operatorname{ad} b)a(x) + \frac{s^2}{2}(\operatorname{ad} b)^2 a(x) + \dots.$$

If (4) is in the normal form up to order $j-1$ solve

$$\pounds_j \tilde{h}_j = f_j - \tilde{g}_j,$$

for $\tilde{h}_j$ and $\tilde{g}_j$ and performs in (4) the transformation

$$x \to \psi_s(x)$$

with $s = 1$ and $\psi_s(x)$ being the flow of

$$\frac{dx}{ds} = \tilde{h}_j(x).$$

$$x = H_m(y) \circ \cdots \circ H_2(y), \qquad H_k = \exp(\tilde{h}_k), \quad k = 2, \dots, m.$$

A grading of the formal vector fields module

Assume that the terms of the function $f(x)$ in (4) depends polynomially on parameters $a_1, \dots, a_\ell$ and let $a = (a_1, \dots, a_\ell)$, so the terms of $f(x) = f(a, x)$ are

$$f^{(i)}_{(\mu_1,\dots,\mu_\ell,\beta_1,\dots,\beta_n)} a_1^{\mu_1} \cdots a_\ell^{\mu_\ell} x_1^{\beta_1} \cdots x_n^{\beta_n}. \tag{10}$$

Example.

$$\begin{aligned}
\dot{x}_1 &= x_1 + a^{(1)}_{10} x_1^2 + a^{(1)}_{01} x_1 x_2 + a^{(1)}_{-13} x_2^3 = x_1(1 + a^{(1)}_{10} x_1 + a^{(1)}_{01} x_2 + a^{(1)}_{-13} x_1^{-1} x_2^3),\\
\dot{x}_2 &= - x_2 + a^{(2)}_{10} x_1 x_2 + a^{(2)}_{01} x_2^2 + a^{(2)}_{02} x_2^3 = x_2(-1 + a^{(2)}_{10} x_1 + a^{(2)}_{01} x_2 + a^{(2)}_{02} x_2^2).
\end{aligned} \tag{11}$$

The normal form of (11) up to order 5 is

$$\dot{x} = \mathrm{diag}(1, -1)x + g_2(x) + g_3(x) + g_4(x) + g_5(x), \tag{12}$$

where $x = (x_1, x_2)^T$, $g_2(x) = g_4(x) = 0$ and

$$\begin{aligned}
g_3(x) &= (x_1(-a^{(1)}_{01} a^{(1)}_{10} + a^{(1)}_{01} a^{(2)}_{10}) x_1 x_2,\ x_2(-a^{(1)}_{01} a^{(2)}_{10} + a^{(2)}_{01} a^{(2)}_{10}) x_1 x_2)^T,\\
g_5(x) &= (x_1((a^{(1)}_{01})^2 a^{(1)}_{10} a^{(2)}_{10} + a^{(1)}_{01} a^{(1)}_{10} a^{(2)}_{01} a^{(2)}_{10} - 2(a^{(1)}_{01})^2 (a^{(2)}_{10})^2) x_1^2 x_2^2,\\
&\qquad x_2(-a^{(1)}_{01} a^{(1)}_{10} a^{(2)}_{01} a^{(2)}_{10} + a^{(1)}_{10} a^{(2)}_{02} a^{(2)}_{10} + 2(a^{(1)}_{01})^2 (a^{(2)}_{10})^2 - a^{(1)}_{01} a^{(2)}_{01} (a^{(2)}_{10})^2 + 2a^{(2)}_{02} (a^{(2)}_{10})^2)
\end{aligned}$$

12 monomials involved parameters, four monomials $x_1^2 x_2$, $x_1 x_2^2$, $x_1^3 x_2^2$, $x_1^2 x_2^3$.

$$\dot{x}_k = \lambda_k x_k + x_k \sum_{\bar{\imath}\in\Omega_k} a_{\bar{\imath}}^{(k)} x^{\bar{\imath}} \qquad (k = 1, \dots, n), \tag{13}$$

Ω_k $(k = 1, \dots, n)$ is a fixed ordered set of multi-indices (n-tuples) $\bar{\imath} = (i_1, \dots, i_n)$, whose k-th entry is from $\mathbb{N}_{-1} = \{-1\} \cup \mathbb{N}_0$ and all other entries are from $\mathbb{N}_0$.
ℓ – the number of parameters $a_{\bar{\imath}}^{(k)}$ in (13).
Let $\tilde{L}$ be the $\ell \times n$ matrix which rows are all n-tuples $\bar{\imath}$

$$\begin{aligned}\dot{x}_1 &= \ \ x_1 + a_{10}^{(1)} x_1^2 + a_{01}^{(1)} x_1 x_2 + a_{-13}^{(1)} x_2^3 = x_1(1 + a_{10}^{(1)} x_1 + a_{01}^{(1)} x_2 + a_{-13}^{(1)} x_1^{-1} x_2^3),\\ \dot{x}_2 &= -\, x_2 + a_{10}^{(2)} x_1 x_2 + a_{01}^{(2)} x_2^2 + a_{02}^{(2)} x_2^3 = x_2(-1 + a_{10}^{(2)} x_1 + a_{01}^{(2)} x_2 + a_{02}^{(2)} x_2^2).\end{aligned}$$

$$\tilde{L} = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 2 \end{pmatrix}^T$$

For $\nu = (\nu_1, \dots, \nu_\ell) \in \mathbb{N}_0^\ell$

$$L(\nu) = \nu \tilde{L}. \tag{14}$$

$L(\nu)$ is the row vector $L(\nu) = (L_1(\nu), \dots, L_n(\nu))$. $L : \mathbb{N}_0^\ell \to \mathbb{Z}^n$.

$$L(\nu) = (\nu_1 - \nu_3 + \nu_4, \nu_2 + 3\nu_3 + \nu_5 + 2\nu_6).$$

• We use $L(\nu)$ to grade the space of parameters ($L(\nu)$ is degree of a monomial).

Denote by a the ordered (according to the order in Ω_k, $k = 1, 2, \ldots, n$) ℓ-tuple of parameters of system (13),

$$a = (a^{(1)}_{\bar{\imath}(1)}, a^{(1)}_{\bar{\imath}(2)}, \ldots, a^{(n)}_{\bar{\imath}(\ell)})$$

by $\mathbb{F}[a]$ the ring of polynomials in variables $a^{(1)}_{\bar{\imath}(1)}, \ldots, a^{(n)}_{\bar{\imath}(\ell)}$ over $\mathbb{F}$. Any monomial in parameters of (13) has the form

$$a^{\nu} = (a^{(1)}_{\bar{\imath}(1)})^{\nu_1}(a^{(1)}_{\bar{\imath}(2)})^{\nu_2} \cdots (a^{(n)}_{\bar{\imath}(\ell)})^{\nu_\ell} \qquad (\nu \in \mathbb{N}_0^{\ell}). \tag{15}$$

Definition 1

For $m \in \mathbb{Z}^n$, a (Laurent) polynomial $p(a)$, $p = \sum_{\nu \in Supp(p)} p^{(\nu)} a^{\nu}$, is an m-polynomial if for every $\nu \in Supp(p) \subset \mathbb{N}_{-1}^{\ell}$, $L(\nu) = m$.

For a given $m \in \mathbb{Z}^n$ let R_m be the subset of $\mathbb{F}[a]$ consisting of all m-polynomials. Let

$$R = \oplus_{m \in \mathbb{Z}^n} R_m.$$

Since

$$R_{m_1} R_{m_2} \subseteq R_{m_1+m_2},$$

R is a graded ring, $R_0 = \mathbb{F}$.
R_m as well as R are vector spaces over $\mathbb{F}$ for the usual addition and the multiplication by numbers from $\mathbb{F}$.

$$\mathcal{M} = \{(m_1, \dots, m_n) \in \mathbb{N}^n_{-1} : |m| \geq 0, m \in \mathrm{im}L, m_j = -1 \text{ for at most one } j\}.$$

Let M_j be the space of vector fields of the form

$$\dot{x}_j = x_j \sum_{\substack{m \in \mathcal{M}, \\ m_i \geq 0 \text{ if } i \neq j}} p_j^{(m)}(a) x^m \qquad (j = 1, \dots, n)$$

where $p_j^{(m)}(a) \in R_m$ for all $j = 1, \dots, n$.
Cf. Usual grading of power series:

$$\dot{x}_j = x_j \sum_{k=1}^{\infty} \sum_{|m|=k} q_j^{(m)}(a) x^m \qquad (j = 1, \dots, n)$$

Let

$$M = M_1 \times \cdots \times M_n.$$

Denoting $p^{(m)}(a) = \sum_{j=1}^n e_j p_j^{(m)}(a)$, for every member of M there exists a finite or infinite set $\mathcal{L} \subset \mathcal{M}$ such that it can be written as

$$\sum_{m \in \mathcal{L}} x \odot p^{(m)}(a) x^m \tag{16}$$

with $\odot$ denoting Hadamard multiplication.
M is an additive group and moreover, M is a module over the ring $(\mathbb{F}^n, +, \odot)$ with the multiplicative unit $\bar{1} = \sum_{i=1}^n e_i \in \mathbb{F}^n$.

Lie brackets in M

Lemma 1

Any element $\mathbf{\Theta} = (\Theta^{(1)}, \ldots, \Theta^{(n)}) \in M$ *can be written in the form*

$$\mathbf{\Theta} = \sum_{\mu \in \omega} (\theta(\mu) \odot x) a^{\mu} x^{L(\mu)}, \qquad (17)$$

where $\theta(\mu) = (\theta_1(\mu), \ldots, \theta_n(\mu))$, ω *is a finite or infinite subset of* $\mathbb{N}_0^{\ell}$ *such that if* $\mu \in \omega$ *then* $L(\mu) \in \mathcal{M}$. *Additionally, if* $L_j(\mu) = -1$, *then* $\theta(\mu) = \theta_j(\mu) e_j$.

NB. In (17) a^{μ} is $L(\mu)$-polynomial.
Lie bracket:

$$[\mathbf{\Theta}, \mathbf{\Phi}] := (D\mathbf{\Phi})\mathbf{\Theta} - (D\mathbf{\Theta})\mathbf{\Phi}$$

for any $\mathbf{\Theta}, \mathbf{\Phi} \in M$.

Lemma 2

If $\mathbf{\Theta} = (\theta \odot x) a^{\mu} x^{L(\mu)}$ *and* $\mathbf{\Phi} = (\phi \odot x) a^{\nu} x^{L(\nu)}$, *where* $\mu, \nu \in \mathbb{N}_0^{\ell}$, $\theta = (\theta_1, \ldots, \theta_n)^T$ *and* $\phi = (\phi_1, \ldots, \phi_n)^T$, *then*

$$[\mathbf{\Theta}, \mathbf{\Phi}] = ((\langle L(\nu), \theta \rangle \phi - \langle L(\mu), \phi \rangle \theta) \odot x) a^{\mu+\nu} x^{L(\mu+\nu)} \in M. \qquad (18)$$

Corollary 1

If $\mathbf{\Theta}, \mathbf{\Phi} \in M$, then $[\mathbf{\Theta}, \mathbf{\Phi}] \in M$. Moreover, if

$$\mathbf{\Phi} = \sum_{\nu \in \omega_1} (\phi(\nu) \odot x) a^{\nu} x^{L(\nu)}, \quad \mathbf{\Theta} = \sum_{\mu \in \omega_2} (\theta(\mu) \odot x) a^{\mu} x^{L(\mu)}$$

then

$$[\mathbf{\Theta}, \mathbf{\Phi}] = \sum_{\mu \in \omega} \sum_{\nu \in \omega_1} ((\langle L(\mu), \phi(\nu) \rangle \theta(\mu) - \langle L(\nu), \theta(\mu) \rangle \phi(\nu)) \odot x) a^{\mu+\nu} x^{L(\mu+\nu)}. \qquad (19)$$

Let $\mathcal{U}_s^{\ell}$, $s \geq 0$, be the space of polynomial vector fields of the form

$$\sum_{|\mu|=s} (\theta(\mu) \odot x) a^{\mu} x^{L(\mu)}. \qquad (20)$$

Then $M = \bigoplus_{s=0}^{\infty} \mathcal{U}_s^{\ell}$.
Cf. $\mathcal{V}_j^n$ is the space of polynomial vector fields of degree j.

$$U_s = \{(\bar{1} \odot x) a^{\mu} x^{L(\mu)} : |\mu| = s\}. \qquad (21)$$

Any vector field $\mathbf{\Theta}_s$ of the form (20) can be written as

$$\mathbf{\Theta}_s = \sum_{|\mu|=s} (\theta(\mu) \odot (x a^{\mu} x^{L(\mu)})) = \sum_{|\mu|=s} (\theta(\mu) \odot x) a^{\mu} x^{L(\mu)},$$

$\Longrightarrow$ $\mathcal{U}_s^{\ell}$ is a module generated by U_s over the ring $(\mathbb{F}^n, +, \odot)$ with the operation of multiplication by the elements of the ring being the Hadamard product.

Definition 2

We say that $\mathbf{\Theta} \in M$ *is of level s if* $\mathbf{\Theta} \in \mathcal{U}_s^\ell$ *and* $\mathbf{\Theta} \in M$ *is of level at least s if each term*

$$\mathbf{\Theta} = (\theta \odot x) a^\mu x^{L(\mu)}$$

of $\mathbf{\Theta}$ *is in some of* $\mathcal{U}_{s+j}^\ell$*, where* $j \in \mathbb{N}_0$*.*

By Lemma 2 if $\mathbf{\Theta} \in \mathcal{U}_s^\ell$, $\mathbf{\Phi} \in \mathcal{U}_t^\ell$ and ad $\mathbf{\Theta}$ is the adjoint operator acting on $\mathbf{\Phi}$ by

$$(\text{ad } \mathbf{\Theta})\, \mathbf{\Phi} = [\mathbf{\Theta}, \mathbf{\Phi}],$$

then $(\text{ad } \mathbf{\Theta})^i \mathbf{\Phi}$ is an element of $\mathcal{U}_{is+t}^\ell$, that is,

$$(\text{ad } \mathbf{\Theta})^i : \mathcal{U}_t^\ell \to \mathcal{U}_{is+t}^\ell. \tag{22}$$

N.B. $(\text{ad } \mathbf{\Theta})^i$ "lifts" the space $\mathcal{U}_t^\ell$ to the space $\mathcal{U}_{is+t}^\ell$.

The normal form algorithm

Any equation of the form (13) can be written as

$$\dot{x} = a_0(x) + a_1(x) + a_2(x) + \cdots = a(x), \tag{23}$$

where $a_0(x) = \lambda \odot x$, $a_s(x) \in \mathcal{U}_s^\ell$ for $s \geq 1$,

$$a_s(x) = \sum_{\mu \in \sigma(s)} (\alpha(\mu) \odot x) a^\mu x^{L(\mu)},$$

where $\alpha(\mu) \in \mathbb{F}^n$. Since $a_0(x) \in \mathcal{U}_0^\ell$ and the nonlinear part $a_1(x)$ of (13) is from $\mathcal{U}_1^\ell$, equation (13) is of the form (23).

$$\begin{aligned}
\dot{x}_1 &= \ \ x_1 + a_{10}^{(1)} x_1^2 + a_{01}^{(1)} x_1 x_2 + a_{-13}^{(1)} x_2^3 = x_1(1 + a_{10}^{(1)} x_1 + a_{01}^{(1)} x_2 + a_{-13}^{(1)} x_1^{-1} x_2^3), \\
\dot{x}_2 &= -\,x_2 + a_{10}^{(2)} x_1 x_2 + a_{01}^{(2)} x_2^2 + a_{02}^{(2)} x_2^3 = x_2(-1 + a_{10}^{(2)} x_1 + a_{01}^{(2)} x_2 + a_{02}^{(2)} x_2^2).
\end{aligned}$$

$$a_0(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\begin{aligned}
a_1(x) = &\begin{pmatrix} 1 \\ 0 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} a_{10}^{(1)} x_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} a_{01}^{(1)} x_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} a_{-13}^{(1)} x_1^{-1} x_2^3 + \\
&\begin{pmatrix} 0 \\ 1 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} a_{10}^{(2)} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} a_{01}^{(2)} x_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} a_{02}^{(2)} x_2^2.
\end{aligned}$$

Lemma 3

If

$$\langle L(\mu), \lambda \rangle = 0, \tag{24}$$

then all entries of the vector field

$$(\alpha(\mu) \odot x) a^{\mu} x^{L(\mu)} \tag{25}$$

are resonant terms.

Lemma 3 justifies the following definition.

Definition 3

It is said that a term of the form (25) *of the right-hand side of* (23) *is resonant if* (24) *holds.*

Definition 4

We say that equation (23) is in the normal form up to level s if all non-resonant terms in $a_1(x), \dots, a_s(x)$ are equal to zero.

If equation (23) is in the normal form for all levels $s \in \mathbb{N}$, then it is in the normal form in the usual sense.
We say that the operator $\pounds^a : M \to M$ which acts on e (17) by

$$\pounds^a(\sum_{\mu\in\omega}(\theta(\mu)\odot x)a^\mu x^{L(\mu)}) = \sum_{\mu\in\omega}\langle L(\mu),\lambda\rangle(\theta(\mu)\odot x)a^\mu x^{L(\mu)}) \tag{26}$$

is the *homological operator* of (23) $\pounds^a$ is a homomorphism of the module M over $\mathbb{F}^n$. Indeed, for any $\phi \in \mathbb{F}^n$

$$\begin{aligned}\pounds^a(\phi\odot\sum_{\mu\in\omega}(\theta(\mu)\odot x)a^\mu x^{L(\mu)}) &= \sum_{\mu\in\omega}\langle L(\mu),\lambda\rangle(\phi\odot\theta(\mu)\odot x)a^\mu x^{L(\mu)} \\ &= \phi\odot\pounds^a(\sum_{\mu\in\omega}(\theta(\mu)\odot x)a^\mu x^{L(\mu)})).\end{aligned}$$

The restriction of $\pounds^a$ on $\mathcal{U}_s^\ell$ is denoted by $\pounds_s^a$. Obviously, $\pounds_s^a : \mathcal{U}_s^\ell \to \mathcal{U}_s^\ell$. From (26) the set U_s defined by (21) is the set of basis eigenvectors of $\pounds_s^a$ and

$$\mathcal{U}_s^\ell = \mathrm{im}\pounds_j^a \oplus \ker \pounds_j^a.$$

Assume that equation (23) is in the normal form up to level $s-1$, $s \geq 1$, that is, for terms of the form (25) appearing in (23) if $|\mu| \leq s-1$ and $\langle L(\mu), \lambda \rangle \neq 0$, then $\alpha(\mu) = 0$. Then the homological equation

$$\pounds_s^a(h_s) = a_s - g_s$$

can be solved for h_s and g_s as follows:

$$h_s(x) = \sum_{|\mu|=s, \langle L(\mu), \lambda \rangle \neq 0} \frac{1}{\langle L(\mu), \lambda \rangle} (\alpha(\mu) \odot x) a^\mu x^{L(\mu)}, \tag{27}$$

$$g_s(x) = \sum_{|\mu|=s, \langle L(\mu), \lambda \rangle = 0} (\alpha(\mu) \odot x) a^\mu x^{L(\mu)}. \tag{28}$$

Theorem 1

Assume that equation (23) *is in the normal form up to level* $s-1$, $s \geq 1$, *and let*

$$H_s(x) = \exp(h_s(x)) \tag{29}$$

where h_s *is defined by* (27). *Then performing the substitution* $y = H_s(x)$ *and changing* y *to* x *we obtain from* (23) *an equation, which the right hand side is from* M *and is in the normal form up to level* s.

Proof.

According to the theorem on Lie series for vector fields after transformation (29) we obtain from (23) the vector field

$$a(x) + (\text{ad } h_s)a(x) + \sum_{i=2}^{\infty} \frac{1}{i!}(\text{ad } h_s)^i a(x). \tag{30}$$

By (22) the last summand is of the level at least $s+1$ and for the first two we have

$$\begin{aligned} a(x) + (\text{ad } h_s)a(x) = a(x) + [h_s(x), a(x)] = \\ a_0(x) + a_1(x) + \cdots + a_{s-1}(x) + a_s(x) + [h_s(x), a_0(x)] + \ldots, \end{aligned}$$

where the dots stand for the terms of level at least $s+1$.
By (18), (27) and (28) $a_s(x) + [h_s(x), a_0(x)] = g_s(x)$. Since by our assumption (23) is in the normal form up to level $s-1$ the equation (30) is in the normal form up to level s. □

Corollary 2

There are polynomial maps $H_1(x), \dots, H_s(x)$, such that equation (23) is transformed to an equation which is in the normal form up to level s by the transformation $y = H_s \circ \cdots \circ H_1$.

Algorithm A. Set $a_0(x) := Ax$, $a_1(x) := F(x)$, $a_k(x) := 0$ for $k = 2, 3, \dots, m$.
For $s = 1, \dots, m$ do the following:
(i) define h_s and g_s by (27) and (28);
(ii) compute

$$b(x) = \sum_{k=0}^{s-1} \sum_{i=1}^{\lfloor \frac{m-k}{s} \rfloor} \frac{1}{i!} (\mathrm{ad}\ h_s(x))^i a_k(x)$$

and write $b(x) = \sum_{i=s+1}^{m} b_i(x)$, where $b_i(x) \in \mathcal{U}_i^{\ell}$;
(iii) $a_s = g_s$, $a_i = b_i(x)$ for $i = s+1, \dots, m$.

Proposition 1

If system (23) is in normal form up to level s then it is in the normal form up to order at least $s+1$.

Generalized formal vector fields

Definition 5

Let α be a map defined on some subset ω of $\mathbb{N}_0^\ell$

$$\alpha : \omega \subset \mathbb{N}_0^\ell \to \mathbb{F}^n,$$

where α assigns to every $\nu \in \omega$ an n-tuple

$$\alpha_\nu = (\alpha_1(\nu), \dots, \alpha_\ell(\nu)).$$

We say that the n-tuple of the formal power series

$$\hat{\alpha} = \sum_{\nu \in \omega} \alpha_\nu a^\nu, \tag{31}$$

where $\omega = Supp(\hat{\alpha})$, is a generalized vector field.

In more details (31) is

$$\hat{\alpha} = \sum_{\nu \in \omega} \alpha_\nu (a^{(1)}_{\bar{\imath}(1)})^{\nu_1} (a^{(1)}_{\bar{\imath}(2)})^{\nu_2} \cdots (a^{(n)}_{\bar{\imath}(\ell_n)})^{\nu_\ell}, \quad \alpha_\nu \in \mathbb{F}^n.$$

(31) is not a vector field in the usual sense – usual vector field is defined assigning to a vector from $\mathbb{F}^\ell$ a vector of the same dimension, but if a series (31) converges it assigns to a point from $\mathbb{F}^\ell$ a vector from $\mathbb{F}^n$.

Denote the set of all formal vector fields defined by (31) by $\mathcal{A}$.
$\mathcal{A}$ is a module over the ring $(\mathbb{F}^n, +, \odot)$.
For any $k \in \mathbb{N}_0$, let $\mathcal{A}_k$ be the subset of all elements of $\mathcal{A}$ of the form

$$\sum_{\mu:|\mu|=k} \alpha_\mu a^\mu.$$

$\mathcal{A}_k$ is a module over the ring $(\mathbb{F}^n, +, \odot)$, $\mathcal{A}$ is a direct sum of $\mathcal{A}_k$, $k = 0, 1, \ldots$.
Recall that we consider M as the direct sum of modules $\mathcal{U}_s^\ell$ over $\mathbb{F}^n$, $s = 0, 1, 2, \ldots$, and define a module homomorphism

$$\mathfrak{T} : \mathcal{A} \to M$$

$$\mathfrak{T}\left(\sum_{\mu\in\omega} \theta_\mu a^\mu\right) = \sum_{\mu\in\omega} (\theta_\mu \odot x) a^\mu x^{L(\mu)}. \qquad (32)$$

$\mathfrak{T}$ is an isomorphism.

The Lie bracket of $\hat{\theta} = \sum_{\mu \in \omega} \theta_\mu a^\mu$ and $\hat{\phi} = \sum_{\nu \in \omega_1} \phi_\nu a^\nu$:

$$[\hat{\theta}, \hat{\phi}] = \mathfrak{T}^{-1}([\mathfrak{T}(\hat{\theta}), \mathfrak{T}(\hat{\phi})]).$$

By (19) and (32)

$$[\hat{\theta}, \hat{\phi}] = \sum_{\mu \in \omega} \sum_{\nu \in \omega_1} \left(\langle L(\mu), \phi_\nu \rangle \theta_\mu - \langle L(\nu), \theta_\mu \rangle \phi_\nu \right) a^{\mu+\nu}. \tag{33}$$

Since

$$\mathfrak{T}\left([\hat{\psi}, [\hat{\theta}, \hat{\phi}]]\right) = [\mathfrak{T}(\hat{\psi}), \mathfrak{T}([\hat{\theta}, \hat{\phi}])] = [\mathfrak{T}(\hat{\psi}), [\mathfrak{T}(\hat{\theta}), \mathfrak{T}(\hat{\phi})]],$$

it is easily seen that for the Lie bracket in $\mathcal{A}$ defined by (33) the Jacobi identity holds, so $\mathcal{A}$ is a Lie algebra and so $\mathfrak{T}$ defines a Lie algebra isomorphism.

Let $\hat{\alpha}$ be the image of the right hand side of

$$\dot{x} = a_0(x) + a_1(x) + a_2(x) + \cdots = a(x),$$

under the isomorphism $\mathfrak{T}^{-1}$:

$$\mathfrak{T}^{-1}(a(x)) = \sum_{k=0}^{\infty} \hat{\alpha}_k,$$

where

$$\hat{\alpha}_0 = \lambda, \quad \hat{\alpha}_k = \sum_{\mu \in \sigma(k)} \alpha_\mu a^\mu \quad \text{for } k \geq 1. \tag{34}$$

Definition 6

It is said that the generalized vector field $\hat{\alpha} = \sum_{k=0}^{\infty} \hat{\alpha}_k$ *(where* $\hat{\alpha}_k$ *is of the form* (34)*) is in the normal form up to level* s *if the coefficients of all non-resonant terms in* $\hat{\alpha}_1, \ldots, \hat{\alpha}_s$ *are equal to zero.*

Assume that $\hat{\alpha}$ is in the normal form up to level $s-1$, $s \geq 1$. Let

$$\hat{\eta}_s = \sum_{\substack{\mu : |\mu| = s, \\ \langle L(\mu), \lambda \rangle \neq 0}} \frac{1}{\langle L(\mu), \lambda \rangle} \alpha_\mu a^\mu, \quad \hat{\zeta}_s = \sum_{\substack{\mu : |\mu| = s, \\ \langle L(\mu), \lambda \rangle = 0}} \alpha_\mu a^\mu. \tag{35}$$

that is $\hat{\eta}_s = \mathfrak{T}^{-1}(h_s(x))$, $\hat{\zeta}_s = \mathfrak{T}^{-1}(g_s(x))$, where $h_s(x)$ and $g_s(x)$ are defined by (27) and (28), respectively.

Algorithm B.
Set $\hat{\alpha}_0 := \lambda$, $\hat{\alpha}_1 := \sum_{\bar{\imath}\in\Omega} e_k a_{\bar{\imath}}^{(k)}$, $\hat{\alpha}_k := 0$ for $k = 2, 3, \dots, m$.
For $s = 1, \dots, m$ do the following:

(i) Define $\hat{\eta}_s$ and $\hat{\zeta}_s$ by (35);

(ii) Compute

$$\hat{\xi} = \sum_{k=0}^{s-1} \sum_{i=1}^{\lfloor \frac{m-k}{s} \rfloor} \frac{1}{i!} (\mathrm{ad}\ \hat{\eta}_s)^i \hat{\alpha}_k$$

(where $\mathrm{ad}\,\hat{\eta}_s := [\hat{\eta}_s, \cdot]$ is the adjoint operator acting on $\mathcal{A}$) and represent $\hat{\xi}$ in the form $\hat{\xi} = \sum_{i=s}^{m} \hat{\xi}_i$, where $\hat{\xi}_i \in \widehat{U}_i^{\ell}$;

(iii) Let $\hat{\alpha}_s = \hat{\zeta}_s$, $\hat{\alpha}_{s+1} = \hat{\xi}_{s+1}, \dots, \hat{\alpha}_m = \hat{\xi}_m$.

The obtained vector field $\hat{\alpha}$ is in the normal form up to level m.

By Proposition 1 if system (23) is in normal form up to level s then it is in the normal form up to order at least $s+1$. When $\hat{\alpha}$ is in the normal form up to level m, a normal form of

$$\dot{x}_k = \lambda_k x_k + x_k \sum_{\bar{\imath}\in\Omega_k} a_{\bar{\imath}}^{(k)} x^{\bar{\imath}} \qquad (k = 1, \ldots, n), \tag{13}$$

can be built up from $\hat{\alpha}$ using the following procedure:

Set $g_k(x) = (0, 0, \ldots, 0)^T$ for $k = 1, \ldots, m$. For $\mu \in \cup_{k=1}^{m} \sigma(k)$ do the following:
if $\alpha_\mu \neq 0$, $|L(\mu)| = k$, then $g_k(x) = g_k(x) + (\alpha_\mu \odot x) a^\mu x^{L(\mu)}$.

By Proposition 1 if system (23) is in normal form up to level s then it is in the normal form up to order at least $s+1$. When $\hat{\alpha}$ is in the normal form up to level m, a normal form of

$$\dot{x}_k = \lambda_k x_k + x_k \sum_{\bar{\imath} \in \Omega_k} a_{\bar{\imath}}^{(k)} x^{\bar{\imath}} \qquad (k = 1, \ldots, n), \tag{13}$$

can be built up from $\hat{\alpha}$ using the following procedure:

Set $g_k(x) = (0, 0, \ldots, 0)^T$ for $k = 1, \ldots, m$. For $\mu \in \cup_{k=1}^{m} \sigma(k)$ do the following:
if $\alpha_\mu \neq 0,\ |L(\mu)| = k$, then $g_k(x) = g_k(x) + (\alpha_\mu \odot x) a^\mu x^{L(\mu)}$.

$$\dot{x} = Ax + \sum_{k=1}^{m} g_k(x)$$

is the normalization of (13) up to order m.

Example. Consider system

$$\begin{aligned}\dot{x}_1 = &\ x_1 + a^{(1)}_{10}x_1^2 + a^{(1)}_{01}x_1x_2 + a^{(1)}_{-13}x_2^3 = x_1(1 + a^{(1)}_{10}x_1 + a^{(1)}_{01}x_2 + a^{(1)}_{-13}x_1^{-1}x_2^3),\\ \dot{x}_2 = &- x_2 + a^{(2)}_{10}x_1x_2 + a^{(2)}_{01}x_2^2 + a^{(2)}_{02}x_2^3 = x_2(-1 + a^{(2)}_{10}x_1 + a^{(2)}_{01}x_2 + a^{(2)}_{02}x_2^2).\end{aligned} \tag{12}$$

For $\mu \in \mathbb{N}_0^\ell$ we will use the abbreviation $[\mu] = [\mu_1, \dots, \mu_\ell] := a^\mu$.
By Proposition 1 in order to compute the normal form of (11) up to order 5 it is sufficient to compute the normal form of (11) up to level 4.
At the level 0 the set $\sigma(0)$ consists of only one vector, $(0, 0, 0, 0, 0, 0)$ with

$$\hat{\alpha}_0 = \alpha_{(0,0,0,0,0,0)} = \begin{pmatrix}1\\-1\end{pmatrix}[0, 0, 0, 0, 0, 0].$$

Passing to the level 1, $\sigma(1)$ is the set of vectors

$$e_1^T, \dots, e_6^T, \tag{36}$$

which form the standard basis of $\mathbb{Z}^6$. The vector field $\hat{\alpha}_1$ is obtained by using the nonlinear terms of (11):

$$\begin{aligned}\hat{\alpha}_1 &= \begin{pmatrix}1\\0\end{pmatrix}[e_1^T] + \begin{pmatrix}1\\0\end{pmatrix}[e_2^T] + \begin{pmatrix}1\\0\end{pmatrix}[e_3^T] + \begin{pmatrix}0\\1\end{pmatrix}[e_4^T] + \begin{pmatrix}0\\1\end{pmatrix}[e_5^T] + \begin{pmatrix}0\\1\end{pmatrix}[e_6^T]\\ &= \begin{pmatrix}1\\0\end{pmatrix}a^{(1)}_{10} + \begin{pmatrix}1\\0\end{pmatrix}a^{(1)}_{01} + \begin{pmatrix}1\\0\end{pmatrix}a^{(1)}_{-13} + \begin{pmatrix}0\\1\end{pmatrix}a^{(2)}_{02} + \begin{pmatrix}0\\1\end{pmatrix}a^{(2)}_{10} + \begin{pmatrix}0\\1\end{pmatrix}a^{(1)}_{01}.\end{aligned}$$

Then by (i) of Algorithm B for $s = 1$ we have $\zeta_1 = \bar{0}$ and

$$\hat{\eta}_1 = \begin{pmatrix}1\\0\end{pmatrix}[e_1^T] - \begin{pmatrix}1\\0\end{pmatrix}[e_2^T] - \frac{1}{4}\begin{pmatrix}1\\0\end{pmatrix}[e_3^T] - \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}[e_4^T] + \begin{pmatrix}0\\1\end{pmatrix}[e_5^T] - \begin{pmatrix}0\\1\end{pmatrix}[e_6^T]$$
$$= \begin{pmatrix}1\\0\end{pmatrix}a_{10}^{(1)} - \begin{pmatrix}1\\0\end{pmatrix}a_{01}^{(1)} - \frac{1}{4}\begin{pmatrix}1\\0\end{pmatrix}a_{-13}^{(1)} - \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}a_{02}^{(2)} + \begin{pmatrix}0\\1\end{pmatrix}a_{10}^{(2)} - \begin{pmatrix}0\\1\end{pmatrix}a_{01}^{(1)}.$$

When we know level s, that is the set $\sigma(s)$, the next level, the set $\sigma(s+1)$, is obtained by adding to the elements of $\sigma(s)$, one of vectors (36).
According to the Algorithm B we have to set

$$\hat{\alpha}_2 = \hat{\alpha}_3 = \hat{\alpha}_4 = (0, 0, \dots, 0)^T. \tag{37}$$

Next we compute $\hat{\xi}_1 + \cdots + \hat{\xi}_4$ according to (ii), that is, we compute the sum

$$\frac{1}{2}(\mathrm{ad}\,\hat{\eta}_1)^2\hat{\alpha}_0 + \frac{1}{3!}(\mathrm{ad}\,\hat{\eta}_1)^3\hat{\alpha}_0 + \frac{1}{4!}(\mathrm{ad}\,\hat{\eta}_1)^4\hat{\alpha}_0 +$$
$$(\mathrm{ad}\,\hat{\eta}_1)\hat{\alpha}_1 + \frac{1}{2}(\mathrm{ad}\,\hat{\eta}_1)^2\hat{\alpha}_1 + \frac{1}{3!}(\mathrm{ad}\,\hat{\eta}_1)^3\hat{\alpha}_1 +$$
$$\hat{\alpha}_2 + (\mathrm{ad}\,\hat{\eta}_1)\hat{\alpha}_2 + \frac{1}{2}(\mathrm{ad}\,\hat{\eta}_1)^2\hat{\alpha}_2 +$$
$$\hat{\alpha}_3 + (\mathrm{ad}\,\hat{\eta}_1)\hat{\alpha}_3 +$$
$$\hat{\alpha}_4.$$

Then, for level 1 we have

$$(\mathrm{ad}\,\hat{\eta}_1)\hat{\alpha}_0 + \hat{\alpha}_1 = \bar{0} \tag{38}$$

and, for level 2 we obtain

$$\hat{\alpha}_2 + (\mathrm{ad}\,\hat{\eta}_1)\hat{\alpha}_1 + \frac{1}{2}(\mathrm{ad}\,\hat{\eta}_1)^2\hat{\alpha}_0. \tag{39}$$

By (37) $\hat{\alpha}_2 = 0$ and for the second term of the sum given above

$$\begin{aligned}(\mathrm{ad}\,\hat{\eta}_1)\hat{\alpha}_1 = &\begin{pmatrix}-2\\0\end{pmatrix}[1,1,0,0,0,0] + \begin{pmatrix}-5/2\\0\end{pmatrix}[1,0,1,0,0,0] + \begin{pmatrix}3/4\\0\end{pmatrix}[0,1,1,0,0,0]+\\ &\begin{pmatrix}1/2\\0\end{pmatrix}[0,1,0,1,0,0] + \begin{pmatrix}2\\-2\end{pmatrix}[0,1,0,0,1,0] + \begin{pmatrix}-3/4\\0\end{pmatrix}[0,0,1,1,0,0]+\\ &\begin{pmatrix}15/4\\-5/4\end{pmatrix}[0,0,1,0,1,0] + \begin{pmatrix}-9/4\\0\end{pmatrix}[0,0,1,0,0,1] + \begin{pmatrix}0\\3\end{pmatrix}[0,0,0,1,1,0]+\\ &\begin{pmatrix}0\\-1/2\end{pmatrix}[0,0,0,1,0,1] + \begin{pmatrix}0\\2\end{pmatrix}[0,0,0,0,1,1].\end{aligned}$$

From (38) we observe that $(\mathrm{ad}\,\hat{\eta}_1)\,\hat{\alpha}_0 = -\hat{\alpha}_1$ and using this in (39) gives

$$
\begin{aligned}
\frac{1}{2}(\mathrm{ad}\ \eta_1)^2\alpha_0 + (\mathrm{ad}\ \eta_1)\alpha_1 = {} & \begin{pmatrix}-1\\0\end{pmatrix}[1,1,0,0,0,0] + \begin{pmatrix}-5/4\\0\end{pmatrix}[1,0,1,0,0,0] \\
& + \begin{pmatrix}3/8\\0\end{pmatrix}[0,1,1,0,0,0] + \begin{pmatrix}1/4\\0\end{pmatrix}[0,1,0,1,0,0] \\
& + \begin{pmatrix}1\\-1\end{pmatrix}[0,1,0,0,1,0] + \begin{pmatrix}-3/8\\0\end{pmatrix}[0,0,1,1,0,0] \\
& + \begin{pmatrix}15/8\\-5/8\end{pmatrix}[0,0,1,0,1,0] + \begin{pmatrix}-9/8\\0\end{pmatrix}[0,0,1,0,0,1] \\
& + \begin{pmatrix}0\\3/2\end{pmatrix}[0,0,0,1,1,0] + \begin{pmatrix}0\\-1/4\end{pmatrix}[0,0,0,1,0,1] \\
& + \begin{pmatrix}0\\1\end{pmatrix}[0,0,0,0,1,1].
\end{aligned}
$$

Continuing the computations following Algorithm B we obtain

$$
\begin{aligned}
\hat{\alpha}_1 &= 0; \\
\hat{\alpha}_2 &= (-a_{01}^{(1)}a_{10}^{(1)} + a_{01}^{(1)}a_{10}^{(2)}, -a_{01}^{(1)}a_{10}^{(2)} + a_{01}^{(2)}a_{10}^{(2)})^T; \\
\hat{\alpha}_3 &= (0, a_{10}^{(1)}a_{02}^{(2)}a_{10}^{(2)} + 2a_{02}^{(2)}(a_{10}^{(2)})^2)^T; \\
\hat{\alpha}_4 &= ((a_{01}^{(1)})^2 a_{10}^{(1)}a_{10}^{(2)} + a_{01}^{(1)}a_{10}^{(1)}a_{01}^{(2)}a_{10}^{(2)} - 2(a_{01}^{(1)})^2(a_{10}^{(2)})^2, \\
&\quad - a_{01}^{(1)}a_{10}^{(1)}a_{01}^{(2)}a_{10}^{(2)} + 2(a_{01}^{(1)})^2(a_{10}^{(2)})^2 - a_{01}^{(2)}(a_{10}^{(2)})^2)^T.
\end{aligned}
$$

so, the normal form up to level 4 is

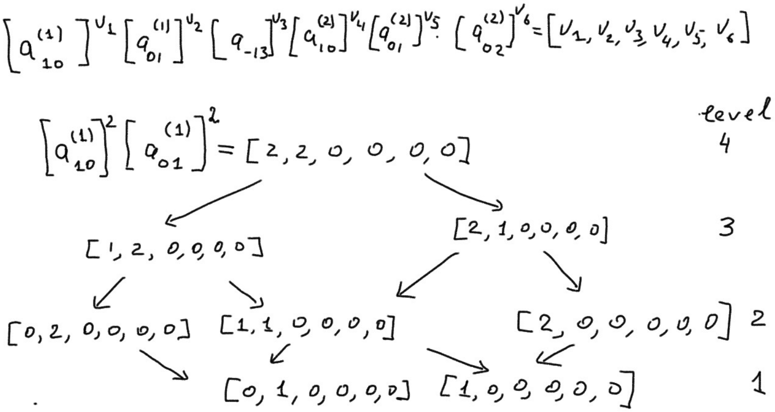
$$
\hat{\alpha} = \hat{\alpha}_0 + \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 + \hat{\alpha}_4.
$$

The above agrees with (12). However, the normal forms differ in the seventh order.

Algorithm B allows parallel computations of terms of normal forms.

Proposition 2

If for some $\kappa \in \mathbb{N}_0^\ell$ we have $\langle L(\kappa), \lambda \rangle = 0$, then the coefficient α_κ in the normal form $\hat{\alpha}$ is computed using only the monomials $\alpha_\mu a^\mu$ such that a^μ divides a^κ, i.e. $\mu_j \le \kappa_j$ for all $j = 1, 2, \ldots, \ell$.



$[\hat{\theta}, \hat{\phi}] = \sum_{\mu \in \omega} \sum_{\nu \in \omega_1} \left(\langle L(\mu), \phi_\nu \rangle \theta_\mu - \langle L(\nu), \theta_\mu \rangle \phi_\nu \right) a^{\mu+\nu}.$

$\hat{\xi} = \sum_{k=0}^{s-1} \sum_{i=1}^{\lfloor \frac{m-k}{s} \rfloor} \frac{1}{i!} (\mathrm{ad}\ \hat{\eta}_s)^i \hat{\alpha}_k$ $(\mathrm{ad}\ \hat{\eta}_s)^i \hat{\alpha}_k : \mathcal{A}_k \to \mathcal{A}_{k+is}$ (that is, "from level k to level $k + is$".

Thus, in order to compute a coefficient α_κ of the normal form we first look for the set of μ involving in the computation of α_κ by means of Algorithm B. Denote this set ω_κ. The set ω_κ can be found using the following procedure:
Let $|\kappa| = s$. Set $p = 1$, $\tau_\kappa(s) = \{\kappa\}$.
While $p < s$ do
set $\tau_\kappa(s-p) = \emptyset$;
for $\mu = (\mu_1, \ldots, \mu_\ell)$:
for $i = 1, \ldots, \ell$: if $\mu_i - i \geq 0$ then $\tau_\kappa(s-p) = \tau_\kappa(k) \cup \{\mu\}$;
set $p = p + 1$.

The output of the procedure are the sets $\tau_\kappa(i)$, $i = 1, \ldots, s-1$, where $\tau_\kappa(i)$ is a subset of elements of level i. Then

$$\tau_\kappa = \cup_{i=1}^{s-1} \tau_\kappa(i)$$

is the subset of $\mathbb{N}_0^\ell$ needed in the computation of α_κ and in order to compute the α_κ one just uses Algorithm B where the Lie brackets are computed with ω_1 and ω_2 in (33) being subsets of τ_κ.

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Thank you for your attention!