Local cyclicity for low degree families of centers

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Main Reference & Grants

The main technique of this talk is detailed in the joint work:

J. Giné, L. F. d. S. Gouveia, J. Torregrosa. Lower bounds for the local cyclicity for families of centers. J. Differential Equations, 275, 309–331, 2021.

Grants

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Local Hilbert Number for cubic and quartic families

Using families of centers and first order Taylor developments:

Theorem (GinGouTor2021)

The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degrees 3 *and* 4 *is (at least)* 12 *and* 21*, respectively.*

Using fixed systems and higher order Taylor developments:

Theorem (GouTor2021,BasBuzTor2021)

The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degrees 5 *and* 6 *is (at least)* 33 *and* 48*, respectively.*

Why have we started with $n = 3$?

Theorem (Bau1953)

The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the class of polynomial vector fields of degree 2 is (exactly) 3.

- Implicit Function Theorem.
- Extensions of Implicit Function Theorem (Weighted Blow-ups).
- Parallelization and higher order developments.
- Working with families instead of fixed vector fields.

We perform a change to polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ on system

$$(\dot{x}, \dot{y}) = (-y + X(x,y), x + Y(x,y)),$$

to obtain

$$(\dot{r}, \dot{\varphi}) = \left(\sum_{k=1}^{\infty} \xi_k(\varphi) r^{k+1}, 1 + \sum_{k=1}^{\infty} \zeta_k(\varphi) r^k \right),$$

where $\xi_k(\varphi)$ and $\zeta_k(\varphi)$ are homogeneous polynomials in $\sin\varphi$ and $\cos\varphi$ of degree $k+2$. Eliminating the time we have

$$\frac{dr}{d\varphi} = \sum_{k=2}^{\infty} R_k(\varphi) r^k,$$

where $R_k(\varphi)$ are 2π-periodic functions of φ and the series is convergent for all φ and for all sufficiently small r.

The Lyapunov constants

The initial value problem with the initial condition $(r, \varphi) = (\rho, 0)$ has a unique solution

$$r(\varphi) = \rho + \sum_{k=2}^{\infty} u_k(\varphi)\rho^k,$$

which is convergent for all $0 \leq \varphi \leq 2\pi$ and all $\rho < r^*$, for some sufficiently small $r^* > 0$. The coefficients $u_k(\varphi)$ can be determined by simple quadratures.

The **Lyapunov constants** are defined as the coefficients, in ρ, of the return map $\Pi(\rho) = r(2\pi)$.

$$L_k = u_{2k+1}(2\pi).$$

Which are the properties of the Lyapunov constants? (Key points?)

$$V_{2k+2} \in \langle V_3, \ldots, V_{2k+1} \rangle = \langle L_1, \ldots, L_k \rangle.$$

Limit cycles and centers

The limit cycles are the isolated fixed points of the return map

$$\Pi(\rho) = r(2\pi)$$

or the isolated zeros of the difference map

$$\Delta(\rho) = \Pi(\rho) - \rho.$$

The origin is a center when $\Pi(\rho) \equiv \rho$ or $\Delta(\rho) \equiv \rho$.

Questions

- How can we use the **constants** for obtaining **oscillations** bifurcating from the origin?
- Are they **independent**?
- Does the number of oscillations depend on the parameters of a **family**?
- How can we make a profit by perturbing families instead of individual vector fields?
- Implicit Function Theorem and its extensions . . ., that is, how can we use the **varieties intersection theory**?

Perturbing a quadratic center exhibiting 3 limit cycles

The system (that is in Q_4 class)

$$\begin{aligned}\dot{x} &= -\,y + 18x^2 + 8xy - 8y^2,\\ \dot{y} &= \;\;x + 4x^2 + 14xy - 4y^2,\end{aligned}$$

has a first integral

$$H = \frac{(80x^3 - 480x^2y + 960xy^2 - 640y^3 + 120xy - 240y^2 - 30y - 1)^2}{(20x^2 - 80xy + 80y^2 + 20y + 1)^3}$$

and the origin is a center.

Perturbing a quadratic center exhibiting 3 limit cycles

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and the origin is a center. In complex coordinates, we can write it as

$$\dot{z} = iz + 10z^2 + 5z\bar{z} + (3+4i)\bar{z}^2 + \lambda_2 iz^2 + \lambda_3 z\bar{z} + \lambda_4\bar{z}^2,$$

where $\lambda_2, \lambda_3, \lambda_4$ are real parameters.

The complete Lyapunov constants

$$
\begin{aligned}
L_1 =& -2\pi(5+\lambda_3)\lambda_2,\\
L_2 =& \frac{2\pi}{3}(5+\lambda_3)(4800\lambda_2+200\lambda_3+8\lambda_2^2+759\lambda_2\lambda_3-25\lambda_2\lambda_4+8\lambda_3^2\\
&+18\lambda_2^3+27\lambda_2\lambda_3^2+3\lambda_2\lambda_3\lambda_4),\\
L_3 =& -\frac{\pi}{18}(5+\lambda_3)(136586875\lambda_2+6542500\lambda_3-67500\lambda_4-876\lambda_3^2\lambda_4\\
&-732150\lambda_2^2+41353200\lambda_2\lambda_3-491150\lambda_2\lambda_4+1216450\lambda_3^2\\
&-24600\lambda_3\lambda_4+938100\lambda_2^3+31300\lambda_2^2\lambda_3-336\lambda_2^2\lambda_4\\
&+4525685\lambda_2\lambda_3^2-35558\lambda_2\lambda_3\lambda_4+69240\lambda_3^3+3816\lambda_2^4\\
&+150504\lambda_2^3\lambda_3-28592\lambda_2^3\lambda_4+2082\lambda_2^2\lambda_3^2+209256\lambda_2\lambda_3^3\\
&+15408\lambda_2\lambda_3^2\lambda_4+1242\lambda_3^4+1944\lambda_2^5+4572\lambda_2^3\lambda_3^2+8\lambda_2^3\lambda_3\lambda_4\\
&+3384\lambda_2\lambda_3^4+1112\lambda_2\lambda_3^3\lambda_4).
\end{aligned}
$$

The linear terms of the Lyapunov constants

$$L_1^{[1]} = -10\pi\lambda_2,$$
$$L_2^{[1]} = 16000\pi\lambda_2 + \frac{2000\pi\lambda_3}{3},$$
$$L_3^{[1]} = -\frac{682934375\pi\lambda_2}{18} - \frac{16356250\pi\lambda_3}{9} + 18750\pi\lambda_4.$$

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The rank of the matrix defined by the coefficients $\lambda_2, \lambda_3, \lambda_4$ of $\{L_1^{[1]}, L_2^{[1]}, L_3^{[1]}\}$ is 3 (maximal).

Consequently, adding the trace, 3 limit cycles bifurcate from the origin.

In case that all the perturbation terms are added, only rank 3 is found.

What can we do for systems of higher degree?

Theorem (LiaTor2015)

For $4 \leq n \leq 13$, equation

$$\dot{z} = iz + z^2 + z^3 + \cdots + z^n + \lambda_1 z + \sum_{k+\ell=2}^{n} \lambda_{k,\ell} z^k \bar{z}^\ell,$$

where $\lambda_1 \in \mathbb{R}$, $\lambda_{k,\ell} \in \mathbb{C}$ are perturbing parameters, has at least $n^2 + n - 2$ small limit cycles bifurcating from the origin.

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where $\lambda_1 \in \mathbb{R}$, $\lambda_{k,\ell} \in \mathbb{C}$ are perturbing parameters, has at least $n^2 + n - 2$ small limit cycles bifurcating from the origin.

For $n = 13$, we need $n^2 + n - 2 = 180$ Lyapunov constants.

We have used paralelization computations for achieving the result.

$C_\ell(3) \geq 11$ with first order terms

Theorem (Chr2006)

From the origin of system

$$\begin{cases} \dot{x} = 10(342+53x)(289x-2112y+159x^2-848xy+636y^2), \\ \dot{y} = 605788x-988380y+432745xy-755568y^2+89888xy^2-168540y^3, \end{cases}$$

bifurcate, at least, 11 limit cycles.

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bifurcate, at least, 11 limit cycles.

Proof.

The system has a center at the origin because it has the first integral

$$H(x,y) = \frac{(xy^2+x+1)^5}{x^3(8xy^5+20xy^3+20y^3+15xy+30y/4+16)^2}.$$

The linear part of the first 11 Lyapunov constants $(L_1, L_2, \ldots, L_{11})$ are linearly independent. Then, adding the trace, we have 11 limit cycles. $\square$

$CD_{31}^{(12)}$ in [Zol1996]

The original 1-parameter family that "generically" has 11 limit cycles bifurcating from the center.

$$H(x,y) = \frac{(xy^2+x+1)^5}{x^3(xy^5+5xy^3/2+5y^3/2+15xy/8+15y/4+a)^2}$$

Remarks

- Depending on a the family can have no centers.
- The above proof of $C_\ell(3) \geq 11$ [Chr2006] was taking $a=2$.
- The existence of special values of a was discovered in [YuTia2014](*).

Perturbing reversible quadratic families

How does the local cyclicity of the origin of a reversible center depend on the parameters?

$$\begin{aligned}
\dot{x} &= -y + \alpha x^2 + \beta y^2 + a_{20}x^2 + a_{11}xy + a_{02}y^2,\\
\dot{y} &= x(1-2y) + b_{20}x^2 + b_{11}xy + b_{02}y^2.
\end{aligned}$$

Perturbing reversible quadratic families

How does the local cyclicity of the origin of a reversible center depend on the parameters?

$$\begin{aligned}\dot{x} &= -y + \alpha x^2 + \beta y^2 + a_{20}x^2 + a_{11}xy + a_{02}y^2,\\ \dot{y} &= x(1-2y) + b_{20}x^2 + b_{11}xy + b_{02}y^2.\end{aligned}$$

Using the usual complex notation, $\dot{z} = \mathrm{i}\, z + Az^2 + Bz\bar{z} + C\bar{z}^2$, the well-known Lyapunov quantities for a quadratic system write as

$$\begin{aligned}L_1 &= -2\pi \operatorname{Im}(AB),\\ L_2 &= -\frac{2\pi}{3}(2A+\bar{B})(A-2\bar{B})\bar{B}C,\\ L_3 &= -\frac{5\pi}{4}(|B|^2-|C|^2)(2A+\bar{B})\bar{B}^2C.\end{aligned}$$

Perturbing reversible quadratic families (2)

So, the first-order Taylor series with respect to the perturbation parameters $(a_{20}, a_{11}, a_{02},\ b_{20}, b_{11}, b_{02})$ allow us to write them as

$$
\begin{aligned}
L_1^{[1]} &= \frac{\pi}{2}\big((\alpha+\beta)a_{11}+2(\beta+1)b_{02}-2(\alpha-1)b_{20}\big),\\
L_2^{[1]} &= \frac{\pi}{48}\big((\alpha+\beta)(5\alpha^2+6\alpha\beta+5\beta^2-8\alpha-16\beta-12)a_{11}\\
&\quad -2(12\alpha^3+15\alpha^2\beta-6\alpha\beta^2-5\beta^3+15\alpha^2+22\alpha\beta+11\beta^2-8\alpha-12\beta-4)b_{02}\\
&\quad -2(\alpha-1)(5\alpha^2-6\alpha\beta-15\beta^2+16\alpha+16\beta+4)b_{20}\big),\\
L_3^{[1]} &= -\frac{5\pi}{1024}(\alpha+3\beta-2)(3\alpha+\beta+2)\big((\alpha+\beta-4)(\alpha+\beta)^2a_{11}\\
&\quad +2(\alpha+\beta)(4\alpha^2-\alpha\beta-\beta^2+3\alpha+3\beta-4)b_{02}+2(\alpha-1)(\alpha+\beta)(\alpha-3\beta+4)b_{20}\big).
\end{aligned}
$$

When $\alpha \neq 1$, using the Implicit Function Theorem, there exists a change of variables in the parameter space that write $L_1 = L_1^{[1]} = u_1$.

Perturbing reversible quadratic families (3)

Hence, under the condition $L_1 = u_1 = 0$ we have

$$L_2^{[1]} = -\frac{\pi}{6}(\alpha-1)(3\alpha+5\beta+2)((\alpha+2)b_{02} + (2-\beta)b_{20}),$$
$$L_3^{[1]} = \frac{5\pi}{128}(\alpha-1)(\alpha+3\,\beta-2)(\alpha+\beta)(3\alpha+\beta+2)((\alpha+2)b_{02} + (2-\beta)b_{20}).$$

Generically, that is when $3\alpha + 5\beta + 2 \neq 0$ (because $\alpha \neq 1$), there exists a local change of coordinates such that $L_2 = u_2$. Under this generic condition, when $L_2 = u_2 = 0$ also $L_3 = 0$ and, consequently, the local cyclicity is only 2. Here we have also used the trace parameter to have a versal unfolding of the weak focus of order 2. This property is used to justify that the reversible family is of codimension 3 and it is commonly named as Q_3^R.

Perturbing reversible quadratic families (4)

It remains a carefully study on the straight line $3\alpha + 5\beta + 2 = 0$.

An extra limit cycle appears.

In fact, on this straight line, as $L_3^{[1]} \neq 0$ when $\beta \notin \{0, -1, 2\}$, we have a weak focus of order three that unfolds 3 limit cycles of small amplitude (because it is simple on the decomposition of $L_2^{[1]}$). Consequently, on the straight line $3\alpha + 5\beta + 2 \neq 0$ the codimension of reversible family is the highest for quadratics, that is 4. Hence, the codimension has increased out of Q_4.

We remark that, after an easy checking, this family only intersects with Q_4 when $(\alpha, \beta) = (-2/3, 0)$.

Cubic vector fields

Theorem (GinGouTor2021)

The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degree 3 is (at least) 12.

$CD_{31}^{(12)}$ in [Zol1996] (Again)

The original 1-parameter family that "generically" 11 limit cycles bifurcate from the center:

$$H(x,y) = \frac{(xy^2+x+1)^5}{x^3(xy^5+5xy^3/2+5y^3/2+15xy/8+15y/4+a)^2}.$$

Remarks

- Depending on a the family can have no centers.
- The first proof of $C_\ell(3) \geq 11$ of [Chr2006] was for $a=2$.
- The existence of special values of a such that $C_\ell(3) \geq 12$.

The proof in four steps

1. For having a center: $32a^2 - 75 > 0$.
2. The key point of the proof:

$$L_k^{[1]} = u_1, k = 1, \ldots, 10,$$
$$L_{11}^{[1]} = g(a) f_0(a) u_{11},$$
$$L_{12}^{[1]} = g(a) f_1(a) u_{11},$$

   f_0 and f_1 are polynomials of degrees 26 and 39 in a^2, g is a rational function (numerator and denominator with degrees 69 and 90 resp. in a^2).

3. Special values of a (simple zeros of f_0):

$$a \in \{\pm 2.019925086, \pm 7.444369217, \pm 15.62631048\}.$$

4. The most important step: Check that other parameters do not affect and only first order analysis via parameters is enough.

Other cubics? Can we increase the degree?

- What about the cyclicity of the other families in [Zol1994]?
- Are there other known families with 11, up to first order analysis [BonSad2008]?
- Can we increase the degree of the polynomial vector field?

Quartic vector fields

Theorem (GinGouTor2021)

The number of limit cycles of small amplitude bifurcating from an equilibrium of monodromic type in the classes of polynomial vector fields of degree 4 *is (at least)* 21.

$C_\ell(4) \geq 21$

Proposition (GinGouTor2021)

Consider the unperturbed system

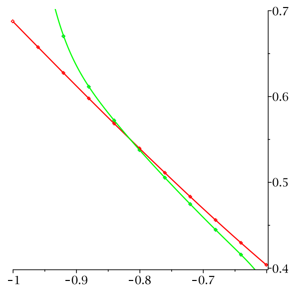
$$\begin{cases} \dot{x} = & -y(1183x^2-68x+1)(1-ax-by), \\ \dot{y} = & (672x^3+1484x^2y-945xy^2-84y^3-58x^2-44xy+30y^2+x)\cdot \\ & (1-ax-by). \end{cases}$$

Then, there exists a pair $(a^*, b^*) \approx (-0.8159251773700849, 0.5506299642821023 9)$ *such that, there exist polynomial perturbations of degree* 4 *such that at least* 21 *limit cycles of small amplitude bifurcate from origin.*

The cubic system without the straight line of equilibria has a center at the origin. Under cubic perturbations unfolds 11 limit cycles, [BonSad2008], and under quartic perturbations (generically) 19.

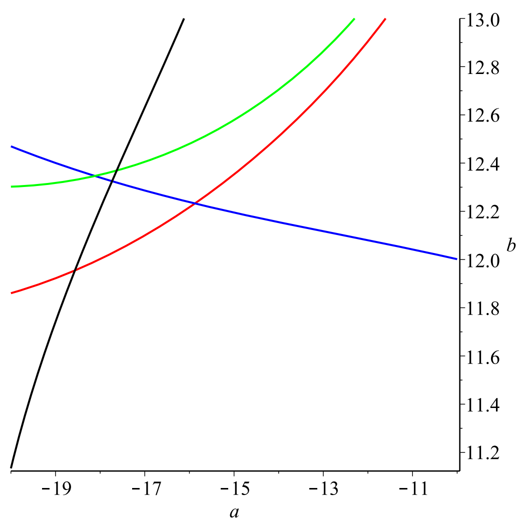
The proof of $C_\ell(21) \geq 21$ (2)

$L_k^{[1]} = u_k, k = 1, \dots, 18.$
$L_{19}^{[1]} = f_0(a,b)u_{19}, \quad L_{20}^{[1]} = f_1(a,b)u_{19}, \quad L_{21}^{[1]} = f_2(a,b)u_{19},$
f_0, f_1, f_2 are polynomials with rational coefficients of degres 180, 182, 184, respectively. The total number of monomials is 16329, 16694, 17063.

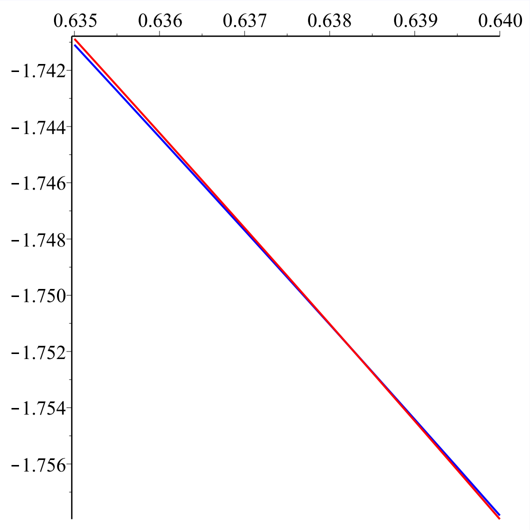


f_0 and f_1 in red and green, respectively.

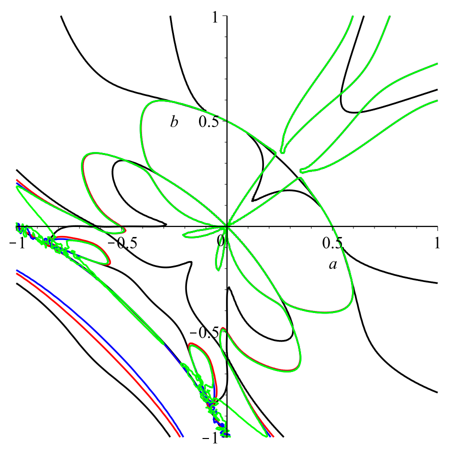
What would we like?

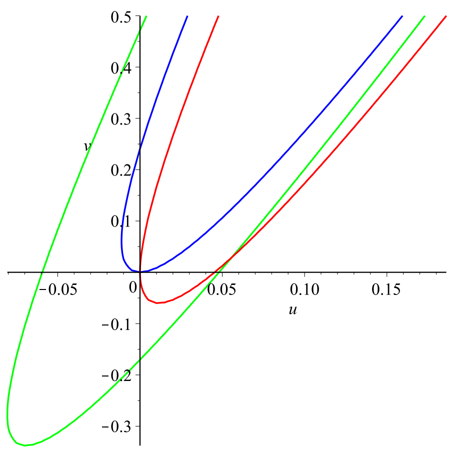


What do we (usually) have locally?



Global plot with a family having two parameters





Higher order developments for sextic vector fields

Proposition (BasBuzTor2021)

The sextic polynomial system

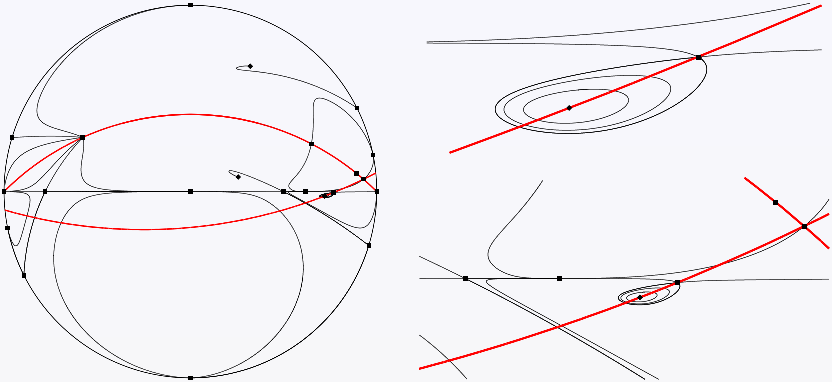
$$
\begin{aligned}
x' = {} & -68082x^6 + 1060844x^5y - 3761510x^4y^2 + 15309875x^3y^3 - 13108500x^2y^4 \\
& + 21847500xy^5 + 487720x^5 - 3970914x^4y - 23536165x^3y^2 + 23595300x^2y^3 \\
& - 135454500xy^4 - 7984x^4 + 4391140x^3y + 61529220x^2y^2 + 307612800xy^3 \\
& - 52434000y^4 - 6983216x^3 - 57185352x^2y - 248187600xy^2 + 104868000y^3 \\
& + 16778880x^2 + 106266240xy + 8389440y^2 - 11185920x - 16778880y, \\
y' = {} & 3y(y-1)(181552x^4 - 784430x^3y + 5386275x^2y^2 - 13108500xy^3 + 21847500y^4 \\
& - 2373680x^3 + 1697310x^2y + 10486800xy^2 - 113607000y^3 + 10158768x^2 \\
& + 48239280xy + 178275600y^2 - 16778880x - 82496160y + 11185920),
\end{aligned}
$$

has a center at the equilibrium point $(3, -1/10)$. *Moreover, perturbing in the class of polynomials of sixth degree, it has local cyclicity at least* 48. *That is* $C_\ell(6) \geq 48$.

Remark

We do not know if a rational first integral exists.

The phase portrait of the sextic center



The proof of the center property

The system of the statement is obtained transforming the vector field

$$X' = \frac{11347}{402410}X^2Y - \frac{49618}{603615}X^2 + \frac{413757}{402410}XY + \frac{17478}{40241}X,$$
$$Y' = \frac{11347}{1207230}XY^2 + Y^3 - \frac{87889}{3621690}XY + \frac{312699}{201205}Y^2 - \frac{1942}{40241}X + \frac{11652}{40241}Y,$$

with the change of coordinates

$$(X, Y) = (x^3/y, x^2/(xy - 5y^2 + 2x + 12y - 4)).$$

The sextic system satisfies an extended reversibility property taking the involution $\varphi(x, y) = (\delta x, \delta^3 y)$ with δ defined implicitly by

$$5y^2\delta^5 + 5y^2\delta^4 - y(x - 5y)\delta^3 - y(x - 5y + 12)\delta^2 + (2x - 4)\delta - 4 = 0.$$

The symmetry line $x - 10y - 4 = 0$ contains the equilibrium point p and it is one of the components of the set $\mathrm{Fix}\,\varphi$.

Proving that $C_\ell(6) \geq 48$

A precise study is necessary because of the degeneracy of the intersection of the varieties (near the origin in the parameters space) defined by the (necessary) Lyapunov constants.

The Taylor series expansions up to first-order of the first Lyapunov constants are written, up to a linear change of coordinates, as $L_k = u_k + O_2(u_1, \ldots, u_{50})$ for $k = 1, \ldots, 43$, and $L_k = O_2(u_1, \ldots, u_{50})$ for $k = 44, \ldots, 50$. Then, using only first-order analysis (adding the trace parameter), only 43 limit cycles of small amplitude bifurcate from the center itself.

Up to a second-order analysis we get $L_k = u_k + O_3(u_1, \ldots, u_{50})$ for $k = 1, \ldots, 43$, and $L_k = O_3(u_1, \ldots, u_{50})$ for $k = 44, \ldots, 50$. So, no more limit cycles bifurcate using $L_1, \ldots, L_{50}$.

Proving that $C_\ell(6) \geq 48$

Up to third-order analysis (and using that the first 43 Lyapunov vanish) we write simplify the study to $L_k = M_k(u_{44}, \ldots, u_{50}) + O_4(u_{44}, \ldots, u_{50})$, for $k = 44, \ldots, 50$, where M_k are homogeneous polynomials of degree 3.

There exists a straight line $u_{44} = -46551/2795\lambda$, $u_{45} = \lambda$, $u_{46} = 13782/2795\lambda$, $u_{47} = 0$, $u_{48} = 0$, $u_{49} = 0$, $u_{50} = 0$ such that, over this line, $M_k = 0$ for $k = 44, \ldots, 50$.

The perturbation $u_{44} = (-46551/2795 + \varepsilon_1)\lambda$, $u_{45} = \lambda$, $u_{46} = (13782/2795 + \varepsilon_2)\lambda$, $u_{47} = \varepsilon_3\lambda$, $u_{48} = \varepsilon_4\lambda$, $u_{49} = \varepsilon_5\lambda$, $u_{49} = \varepsilon_6\lambda$, $u_{50} = \varepsilon_7\lambda$, provides a Jacobian matrix of M_k, for $k = 44, \ldots, 48$, with respect to ε_j, for $j = 1, \ldots, 7$, having rank 5.

The second part of the statement follows because we have a variety of weak-foci of order $43 + 5 = 48$ that unfolds 48 limit cycles of small amplitude bifurcating from the origin.

Final remarks on the proof of $C_\ell(6) \geq 48$

We have computed two Lyapunov constants more. But as the last rank is only 5 we can not obtain a better lower bound for the local cyclicity.

Up to Taylor series of third-order we have checked that both L_{49} and L_{50} vanish.

We have not gone further in the computations of higher-order because of the difficulties and the fact that we have used almost all the perturbation parameters, which are 50 for this system of degree 6.

We have obtained the number of small limit cycles that we think will be the maximum for degree 6 polynomial vector fields ($6^2 + 3 \cdot 6 - 6 = 48$).

Surprisingly, the described mechanism to get new centers has been very interesting in finding centers with very high local cyclicity. Unfortunately the calculations are very hard.

Taking into account the number of free parameters and the fact that the normal form is invariant by a rotation and a rescaling, we can remove two more parameters but we need to add the trace parameter, so the number of essential parameters will provide the total number of limit cycles of small amplitude. So, for $n \geq 3$,

$$C_\ell(n) = n^2 + 3n - 6?$$

This value is our best candidate. At least, although the unique way to study this problem is through the Taylor series of the return map near the origin (computing the Lyapunov constants), because we can obtain only one limit cycle with each parameter (or each Lyapunov constant).

The best values for $C_\ell(n)$ for $n = 2, 3, 4, 5, 6$.

- $C_\ell(2) = 3$ (Bautin 1954) $[n^2 + 3n - 6 = 4]$
- $C_\ell(3) \geq 12$ (Giné, Gouveia, Torregrosa, 2021) $[n^2 + 3n - 6 = 12]$
- $C_\ell(4) \geq 21$ (Giné, Gouveia, Torregrosa, 2021) $[n^2 + 3n - 6 = 22]$
- $C_\ell(5) \geq 33$ (Gouveia, Torregrosa, 2021) $[n^2 + 3n - 6 = 34]$
- $C_\ell(6) \geq 48$ (Bastos, Buzzi, Torregrosa, 2021) $[n^2 + 3n - 6 = 48]$

Another good example in other (but similar) context

Theorem (SanTor2021)

Let $a \in \mathbb{R} \setminus \{0\}$. Consider the 1-parameter family of cubic (holomorphic) reversible systems

$$\dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az)\,.$$

The number of critical periods bifurcating from the origin when perturbing in the class of reversible cubic systems is at least 5 for $a \in \{-3/2, -1, -2/3, 1/2, 2\}$ and 4 otherwise.

With the change $z = x + \mathrm{i}\, y$ we can write the above equation as a planar polynomial differential equation:

$$(\dot{x}, \dot{y}) = \left(-y + 2(a+1)xy - a(3x^2y - y^3), x - (a+1)(x^2 - y^2) + a(x^3 - 3xy^2)\right).$$

Time function (for centers)

Using the solution $r(\varphi)$ into the second equation of the original equation in polar coordinates, obtaining

$$\dot{\varphi} = \frac{d\varphi}{dt} = 1 + \sum_{k=1}^{\infty} F_k(\varphi)\rho^k.$$

Rewriting this equation as

$$dt = \frac{d\varphi}{1 + \sum\limits_{k=1}^{\infty} F_k(\varphi)\rho^k} = \left(1 + \sum_{k=1}^{\infty} \Psi_k(\varphi)\rho^k\right) d\varphi$$

and integrating, we get

$$t - \varphi = \sum_{k=1}^{\infty} \theta_k(\varphi)\rho^k,$$

where $\theta_k(\varphi) = \int_0^{\varphi} \Psi_k(\psi)\, d\psi$. All the series converge for $0 \leq \varphi \leq 2\pi$ and sufficiently small $\rho \geq 0$.

The period constants

The coefficients of the Taylor series, in ρ, of the period function of any closed trajectory define the **period constants**:

$$T_k = \mathcal{T}_{2k+2} = \theta_{2k+2}(2\pi) = \int_0^{2\pi} \Psi_{2k+2}(\psi)\, d\psi,$$

$$\mathcal{T}_{2k+1} \in \langle \mathcal{T}_2, \ldots, \mathcal{T}_{2k} \rangle = \langle T_1, \ldots, T_{k-1} \rangle.$$

Proving the (criticality) cubic result (1)

When $a \in \mathbb{R} \setminus \{-1, 0, 1/2, 2\}$, the rank of the linear developments of first four period constants of this system with respect to $(r_{11}, r_{02}, r_{21}, r_{12})$ is 4. After using the Implicit Function Theorem, the period constants take the form

$$T_k = u_k, \text{ for } k = 1, \ldots, 4.$$

Taking $u_1 = u_2 = u_3 = u_4 = 0$ and $r_{03} = u_5$, the fifth and sixth period constants take the form

$$\begin{aligned} T_5 &= \frac{5}{24} \frac{P(a)}{3a^2 + 2a + 3} u_5 + u_5^2 \sum_{j=0}^{\infty} f_j(a)\, u_5^j, \\ T_6 &= -\frac{1}{42} \frac{Q(a)}{3a^2 + 2a + 3} u_5 + u_5^2 \sum_{j=0}^{\infty} g_j(a)\, u_5^j, \end{aligned} \tag{1}$$

where $P(a) = a^3(a-2)(3a+2)(2a+3)(2a-1)$,
$Q(a) = a^3(a-2)(2a-1)(834a^2 + 1735a + 834)(a+1)^2$, and f_j and g_j are rational functions.

Proving the (criticality) cubic result (2)

Then, we have 4 critical periods when $P(a) \neq 0$ and 5 when $P(a) = 0$, $P'(a) \neq 0$, and $Q(a) \neq 0$. Then, as $a \neq 0$, the statement follows except for the remaining cases $a \in \{-1, 1/2, 2\}$. These cases need more accurate analysis and higher order Taylor developments.

Local critical periods for quartics

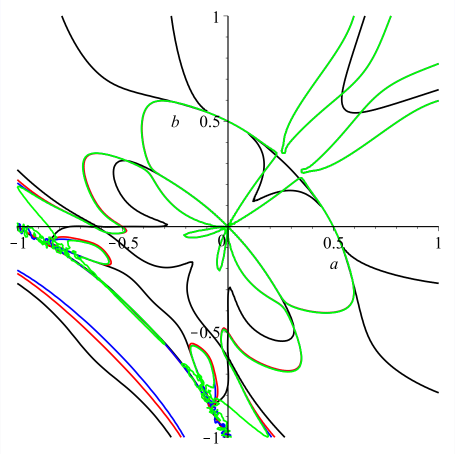
Theorem (SanTor2021)

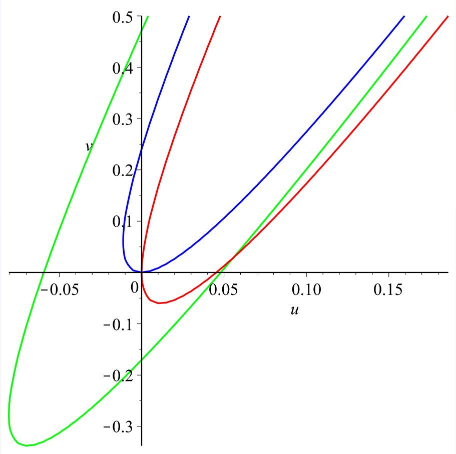
Let $a, b \in \mathbb{R}$. Consider the 2-parameter family of quartic (holomorphic) reversible systems

$$\dot{z} = \mathrm{i}\, z\,(1 - z)\,(1 - az)\,(1 - bz)\,.$$

Generically, at least 8 critical periods bifurcate from the origin when perturbing in the class of reversible quartic centers. Moreover, in this perturbation class there exists a point (a, b) such that at least 10 critical periods bifurcate from the origin.

The plots of the key functions in the plane (a, b)





Can we prove that the lines intersect transversally?
(Poincaré-Miranda Theorem + Computer Assisted Proof)

That's all

Thanks!