Rational solutions and rational limit cycles of Abel Differential Equations

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Abel equation

We consider Abel differential equations of the form

$$x' = A(t)x^3 + B(t)x^2,$$

where A and B are polynomials or trigonometric polynomials.

Two important problems in this setting are

1. Smale–Pugh problem of bounding the number of limit cycles. Lins Neto proved that there is no upper bound on the number of limit cycles of for A, B polynomials or trigonometric polynomials.
2. Poincaré Centre-Focus Problem, which asks when the solutions of (3) in a neighbourhood of the solution $x(t) \equiv 0$ are all closed. This problem for the Abel equation was proposed by Briskin, Françoise, and Yondim.

Polinomial limit cycles

A natural problem in this context is to study polynomial or trigonometric polynomial solutions.

- Gine, Grau and Llibre, in 2011 showed that

$$x' = a_0(t) + a_1(t)x + \cdots + a_n(t)x^n$$

  has at most n polynomial solutions when $a_i(t)$ are polynomials.

- Gasull, Torregrosa and Zhang in 2016 studied

$$a(t)x' = b_0(t) + b_1(t)x + b_2(t)x^2.$$

  If a, b_0, b_1, b_2 are polynomials of degree $n \geq 1$, then it has at most $n+1$ polynomial solutions,
  If a, b_0, b_1, b_2 are trigonometric polynomials of degree $n \geq 1$, then it has at most $2n$ trigonometric polynomials solutions.

Polinomial limit cycles

Other results by

- Cima, Gasull, Mañosas, for Bernoulli or Abel differential equations in 2017.
- Llibre, Valls, for Abel differential equations in 2017.
- Valls, for Abel differential equations in 2017.
- Oliveira, Valls, for Abel differential equations in 2020.

We consider

$$x' = A(t)x^3 + B(t)x^2$$

and study rational solutions for A, B polynomial or trigonometric polynomials.

Polynomial coefficients

Now, we consider

$$x' = A(t)x^3 + B(t)x^2$$

when A and B are polynomials. A natural question is to bound the number of rational solutions in terms of the degree of A, B.

This problem have been studied for rational limit cycles instead of rational solutions, that is, solutions x defined in $[0, 1]$ such that $x(0) = x(1)$ and isolated from other closed solutions.

Liu, Li, Wang, Wu obtained in 2018 examples with at least two rational limit cycles. The problem has also been studied by Llibre and Valls.

Rational solutions

Fix $A, B \in \mathbb{C}[t]$, a *rational solution* is a solution of the form $x(t) = q(t)/p(t)$, where $p, q \in \mathbb{C}[t]$, $p \notin \mathbb{C}$.

Theorem

If $n := \deg(A)$ is even or $\deg(B) > (n-1)/2$ then Abel equation has at most two rational solutions.
If n is odd then

$$\#\{Rational\ solutions\} \leq \binom{n}{(n+1)/2} + 1.$$

This upper bound is not sharp.

Invariant curves

A rational solution q/p is equivalent to a invariant curve of degree one in x, i.e., a curve of the form

$$p(t)x + q(t) = 0.$$

Using Darboux's theory of integrability, following Gine-Santallusia 2010, we study when there exists a first integral of the form

$$f(t,x) := x^{\alpha_0} \prod_{i=1}^{r} (1 + p_i(t)x)^{\alpha_i}.$$

Theorem

If the number of rational solutions, r, is greater than or equal to $(n+1)/2$ then the equation admits a Darboux first integral.

Sketch of the proofs

Proposition (Liu-Li-Wang-Wu 2018)

$q(t)+p(t)x=0$ is an invariant curve of the Abel equation if and only if $q(t)$ is a constant $c\in\mathbb{C}\setminus\{0\}$ and

$$p(t)p'(t)-c\,p(t)B(t)+c^2A(t)=0.$$

Corollary

If $1+p(t)x=0$ is an invariant curve of $x'=A(t)x^3+B(t)x^2$, then p divides A.

Sketch of the proofs

If $1 + p(t)x = 0$ is an invariant curve of the Abel equation, as $p(t)$ divides $A(t)$, then there must exist $r \in \mathbb{C}[t]$ such that

$$A(t) = p(t)r(t),$$

and then

$$p(t)p'(t) - p(t)B(t) + A(t) = 0,$$

transforms into

$$B(t) = p'(t) + r(t). \tag{2.1}$$

Sketch of the proofs

Now suppose $1 + p_1(t)x = 0$, $1 + p_2(t)x = 0$, with $p_1, p_2 \in \mathbb{C}[t]$, are different invariant curves of the Abel equation. Then, if $n = \deg(A)$,

Proposition

$$n + 1 = \deg(p_1) + \deg(p_2)$$

Then $\deg(p_1) = \deg(p_2)$ *if and only if* $\deg(p_1) = (n + 1)/2$.

Corollary

If equation $x' = A(t)x^3 + B(t)x^2$ *has three invariant curves, then they are all of degree* $(n + 1)/2$.

Sketch of the proofs

Proposition

If $1 + p(t)x = 0$ *is an invariant curve of the Abel equation, then* $\deg(p) = (\deg A + 1)/2$ *if and only if* $\deg(B) \leq (n-1)/2$.

Theorem

The Abel equation $x' = A(t)x^3 + B(t)x^2$ *has at most two invariant curves if one of the following conditions hold*

- $\deg(A)$ *is even*
- $\deg(B) > (n-1)/2$

Sketch of the proofs

Proposition

The Abel equation $x' = A(t)x^3 + B(t)x^2$ has at most two invariant curves whose polynomial coefficients of x are proportional.

Theorem

$\binom{n}{(n+1)/2} + 1$ *is an upper bound for the number of invariant curves of the Abel equation.*

Sketch of the proofs

Proposition

Assume that the Abel equation has the invariant curves $1 + p_i(t)x = 0$, $i = 1, \dots, r$. Let $\alpha_i \in \mathbb{C}$, $i = 1, \dots, r$, and $\alpha_0 = -\sum_i^r \alpha_i$. Then $f(t,x) = x^{\alpha_0} \prod_{i=1}^r (1 + p_i(t)x)^{\alpha_i}$ is a Darboux first integral of the equation if and only if

$$\sum_{i=1}^{r} \alpha_i \frac{A(t)}{p_i(t)} = 0.$$

Theorem

Let $n \geq 3$. If the Abel equation has more than $(n+1)/2$ invariant curves then the equation has a Darboux first integral.

Computational results

Low degree cases:

n	$(n+1)/2$	$\binom{n}{(n+1)/2}+1$	Exhaustive bound
1	1	2	2
3	2	4	4
5	3	11	5
7	4	36	-

Exhaustive bound means the optimal upper bound on the number of rational solutions, computed with Singular.

Trigonometric polynomial coefficients

Now, we consider

$$x' = A(t)x^3 + B(t)x^2$$

when A and B are trigonometric polynomials.

A rational limit cycle is a solution of the form $x(t) = Q(t)/P(t)$, where $P(t)$ and $Q(t)$ are real trigonometric polynomials and $P(t) \neq 0$ for all $t \in \mathbb{R}$, such that it is a limit cycle, that is, there is no other periodic solution in a neighbourhood of it.

Theorem

If the degree of $A(t)$ is odd or less than twice the degree of $B(t)$, then Abel equation has at most two non-trivial rational limit cycles. Otherwise, the number of non-trivial rational limit cycles is at most the degree of $A(t)$ plus one.

Sketch of the proof

Let $P(t)$ and $Q(t)$ be real trigonometric polynomials.

Assume $Q(t) - P(t)x = 0$ is an invariant curve and $P(t) \neq 0$ for all $t \in \mathbb{R}$.

It is possible to assume that $Q(t) - P(t)x$ is irreducible in $\mathbb{R}[\cos(t), \sin(t)][x]$ and in $\mathbb{C}[\cos(t), \sin(t)][x]$. Therefore, we will always assume so.

Proposition

If $P(t) \neq 0$ for all $t \in \mathbb{R}$, then $x(t) = Q(t)/P(t)$ is a solution if and only if $Q(t) - P(t)x = 0$ is an invariant curve.

Sketch of the proof

Proposition

The curve $Q(t) - P(t)x = 0$ is an invariant curve if and only if $Q(t) = c \in \mathbb{R}$ and there exists a trigonometric polynomial $R(t)$ such that

$$A(t) = (P(t)/c)R(t), \quad B(t) = -P'(t)/c - R(t).$$

In this case, the corresponding cofactor is equal to

$$A(t)x^2 - (P'(t)/c)x.$$

Sketch of the proof

Proposition

If $1 - P_1(t)x = 0$ *and* $1 - P_2(t)x = 0$ *are two different invariant curves, then* $\deg(P_1) + \deg(P_2) = \deg(A)$.

Corollary

If the Abel equation has three or more non-trivial invariant curves, then they all have degree $\deg(A)/2$.

Corollary

If $\deg(A)$ *is odd or if* $\deg(B) > \deg(A)/2$*, then the Abel equation has at most two non-trivial invariant curves.*

Sketch of the proof

Proposition

Let $\alpha_i \in \mathbb{R}, i = 1, \ldots, r,$ *and* $\alpha_0 := -\sum_{i=1}^{r} \alpha_i$*.*

If $1 - P_i(t)x = 0, i = 1, \ldots, r$ *are invariant curves of* (3)*, then*

$$x^{\alpha_0} \prod_{i=1}^{r} (1 - P_i(t)x)^{\alpha_i}$$

is a first integral if and only if

$$\sum_{i=1}^{r} \alpha_i \frac{A(t)}{P_i(t)} = 0.$$

Thank you!