Polyhedral Neighborhoods vs Tubular Neighborhoods:

New Insights for the Fractal Zeta Functions

Joint work with Michel L. Lapidus

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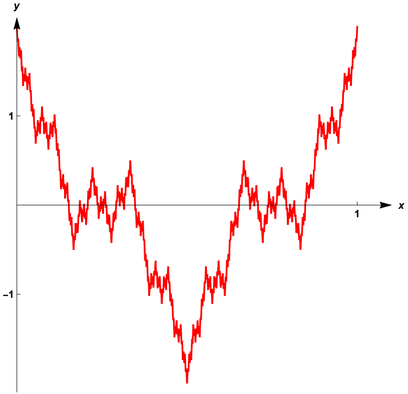
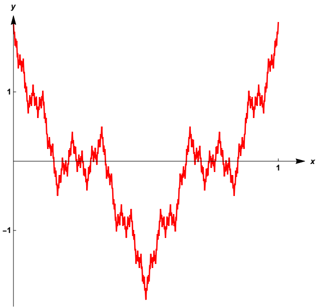
Introduction

A pathological object





$$x \in \mathbb{R} \mapsto \mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(\pi\, b^n\, x\right) \quad , \quad 0 < \lambda < 1 \quad , \quad \lambda\, b > 1$$

Continuous everywhere, while being nowhere differentiable[I],[II].

[I] Karl Weierstrass. "Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differential quotienten besitzen". In: *Journal für die reine und angewandte Mathematik* 79 (1875), pp. 29–31.

[II] Godfrey Harold Hardy. "Weierstrass's Non-Differentiable Function". In: *Transactions of the American Mathematical Society* 17.3 (1916), pp. 301–325.

Minkowski Dimension[III],[IV],[V],[VI]:

$$D_{\mathscr{W}} = 2 + \frac{\ln \lambda}{\ln b} = 2 - \ln_b \frac{1}{\lambda}$$

[III] James L. Kaplan, John Mallet-Paret, and James A. Yorke. "The Lyapunov dimension of a nowhere differentiable attracting torus". In: *Ergodic Theory and Dynamical Systems* 4 (1984), pp. 261–281.

[IV] Feliks Przytycki and Mariusz Urbański. "On the Hausdorff dimension of some fractal sets". In: *Studia Mathematica* 93.2 (1989), pp. 155–186.

[V] Tian-You Hu and Ka-Sing Lau. "Fractal Dimensions and Singularities of the Weierstrass Type Functions". In: *Transactions of the American Mathematical Society* 335.2 (1993), pp. 649–665.

[VI] Claire David. "Bypassing dynamical systems: A simple way to get the box-counting dimension of the graph of the Weierstrass function". In: *Proceedings of the International Geometry Center* 11.2 (2018), pp. 1–16. URL: https://journals.onaft.edu.ua/index.php/geometry/article/view/1028.

An open problem[VII]:

⇝ **Is $D_{\mathscr{W}}$ a Complex Dimension?**

⇝ **What are the Complex Dimensions?**

[VII] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

The Theory of Complex Dimensions:

[VIII], [IX], [X], [XI], [XII]

A natural and intuitive way to characterize fractal strings or drums,

in relation with their intrinsic vibrational properties.

[VIII] Michel L. Lapidus and Machiel van Frankenhuijsen. *Fractal Geometry and Number Theory: Complex Dimensions of Fractal Strings and Zeros of Zeta Functions*. Birkhäuser Boston, Inc., Boston, MA, 2000, pp. xii+268.

[IX] Michel L. Lapidus and Machiel van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*. Springer Monographs in Mathematics. Springer, New York, second revised and enlarged edition (of the 2006 edition), 2013, pp. xxvi+567.

[X] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

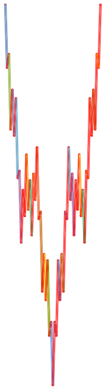
[XI] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. "Distance and tube zeta functions of fractals and arbitrary compact sets". In: *Advances in Mathematics* 307 (2017), pp. 1215–1267.

[XII] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. "Fractal tube formulas for compact sets and relative fractal drums: Oscillations, complex dimensions and fractality". In: *Journal of Fractal Geometry. Mathematics of Fractals and Related Topics* 5.1 (2018), pp. 1–119.

This means:

studying the oscillations of a small neighborhood of the boundary,

where points are located within an epsilon distance from any edge.



Difficulty

When nonlinear and noncontractive IFS are involved

Tubular neighborhoods can only be determined

for the prefractal approximations.

The main question:

Can we pass to the limit?

I. The Geometric Framework

We hereafter place ourselves in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x, y). The horizontal and vertical axes will be respectively refered to as $(x'x)$ and $(y'y)$.

Notation

In the following, λ and N_b are two real numbers such that:

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 \cdot$$

We consider the *Weierstrass function* $\mathscr{W}$, defined, for any real number x, by

$$\mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \cdot$$

Associated graph: **the Weierstrass Curve**.

Due to **the one-periodicity** of the $\mathscr{W}$ function, we restrict our study to the interval $[0, 1[$.

Minkowski (or box-counting) Dimension

$$D_{\mathscr{W}} = 2 + \frac{\ln \lambda}{\ln N_b},$$ *equal to its Hausdorff dimension* [XIII], [XIV], [XV], [XVI]

[XIII] James L. Kaplan, John Mallet-Paret, and James A. Yorke. "The Lyapunov dimension of a nowhere differentiable attracting torus". In: *Ergodic Theory and Dynamical Systems* 4 (1984), pp. 261–281.

[XIV] Krzysztof Barańsky, Balázs Bárány, and Julia Romanowska. "On the dimension of the graph of the classical Weierstrass function". In: *Advances in Mathematics* 265 (2014), pp. 791–800.

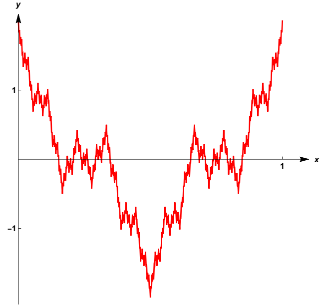
[XV] Weixiao Shen. "Hausdorff dimension of the graphs of the classical Weierstrass functions". In: *Mathematische Zeitschrift* 289 (1-2 2018), pp. 223–266.

[XVI] Gerhard Keller. "A simpler proof for the dimension of the graph of the classical Weierstrass function". In: *Annales de l'Institut Henri Poincaré – Probabilités et Statistiques* 53.1 (2017), pp. 169–181.

The Weierstrass Curve as a Cyclic Curve

In the sequel, we identify the points

$$(0, \mathscr{W}(0)) \quad \text{and} \quad (1, \mathscr{W}(1)) = (1, \mathscr{W}(0)) \cdot$$



Remark

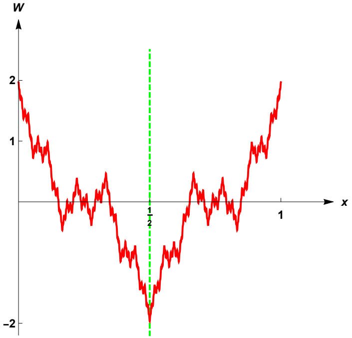
The above convention makes sense, in so far as the points $(0, \mathscr{W}(0))$ and $(1, \mathscr{W}(1))$ have **the same vertical coordinate**, in addition to the periodic properties of the $\mathscr{W}$ function.

Property (Symmetry with respect to the vertical line $x = \frac{1}{2}$)

Since, for any $x \in [0,1]$*:*

$$\mathscr{W}(1-x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\pi N_b^n - 2\pi N_b^n x\right) = \mathscr{W}(x)$$

the Weierstrass Curve is **symmetric with respect to the vertical straight line** $x = \frac{1}{2}$.



Proposition (Nonlinear and Noncontractive Iterated Function System (IFS))

We approximate the restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the Weierstrass Curve, by a sequence of finite graphs, built through an iterative process, by using **the nonlinear iterated function system** *(IFS) of the family of C^∞ maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ denoted by*

$$\mathscr{T}_{\mathcal{W}} = \left\{ T_0, \cdots, T_{N_b-1} \right\} ,$$

where, for $0 \leq i \leq N_b - 1$ and any point (x,y) of $\mathbb{R}^2$,

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\,\pi \left(\frac{x+i}{N_b} \right) \right) \right) \cdot$$

Property (Attractor of the IFS)

The Weierstrass Curve is **the attractor** *of the IFS $\mathscr{T}_{\mathcal{W}}$:* $\Gamma_{\mathcal{W}} = \displaystyle\bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$.

Fixed Points

For any integer i belonging to $\{0, \cdots, N_b - 1\}$, we denote by:

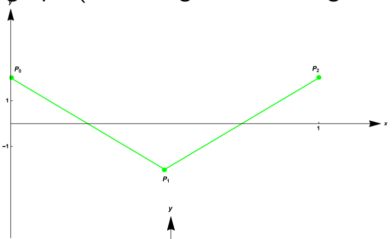
$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\,\pi\, i}{N_b - 1} \right) \right)$$

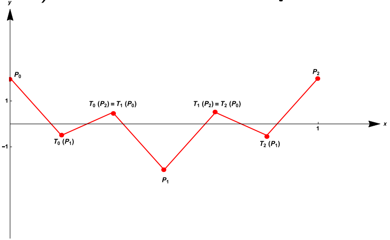
the fixed point of the map $\boldsymbol{T_i}$.

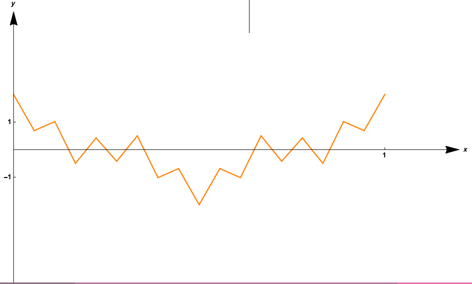
Sets of vertices, Prefractals

We set: $\boldsymbol{V}_0 = \left\{\boldsymbol{P}_0, \cdots, \boldsymbol{P}_{\boldsymbol{N_b}-1}\right\}$, and, for any $m \in \mathbb{N}^\star$: $\boldsymbol{V_m} = \bigcup\limits_{i=0}^{N_b-1} \boldsymbol{T_i}\left(\boldsymbol{V_{m-1}}\right)$.

For $m \in \mathbb{N}$, the set of points V_m, where two consecutive points are linked, is an oriented graph (according to increasing abscissa): the $\boldsymbol{m^{th}}$**-order** $\mathscr{W}$**-prefractal** $\Gamma_{\mathscr{W}_m}$.

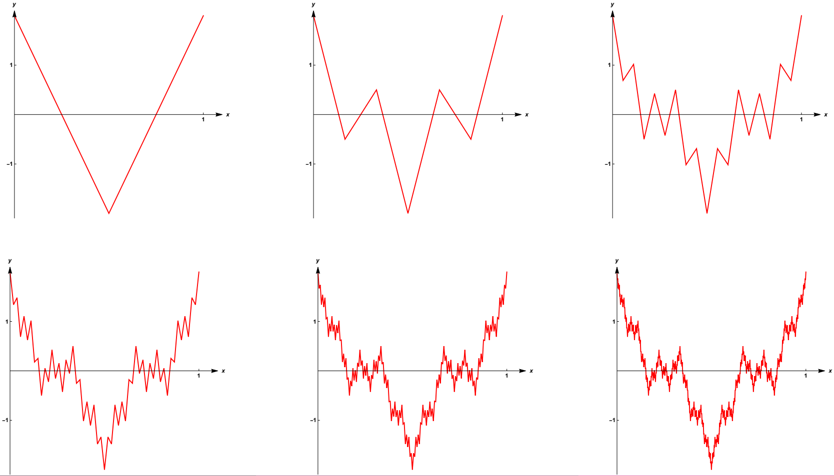






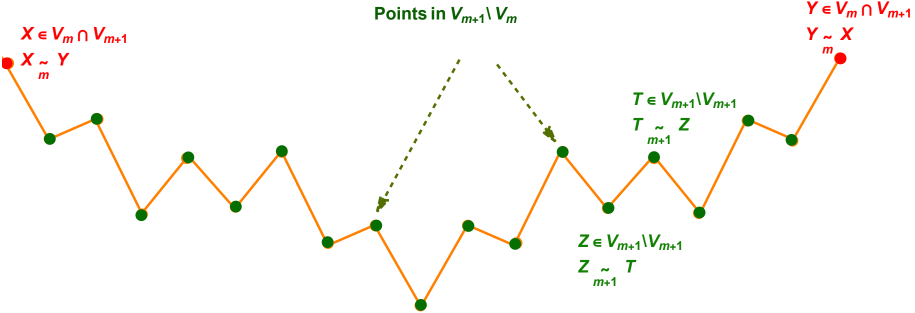
The Weierstrass IFD

We call **Weierstrass Iterated Fractal Drums (IFD)** the sequence of prefractal graphs which converge to the Weierstrass Curve.



Adjacent Vertices, Edge Relation

For any natural integer m, the prefractal graph $\Gamma_{\mathscr{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$: two vertices X and Y of $\Gamma_{\mathscr{W}_m}$, i.e. two points belonging to V_m, will be said to be **adjacent** (i.e., neighboring or junction points) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathscr{W}_m}$; we then write $X \underset{m}{\sim} Y$. **This edge relation depends on m**, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.

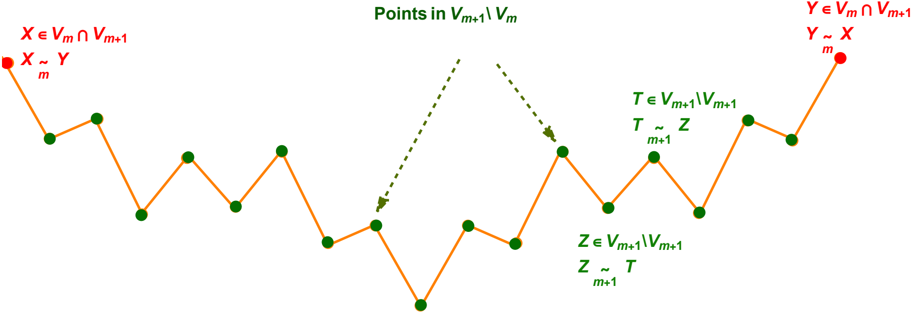


Property

For any natural integer m, we have that

i. $V_m \subset V_{m+1}$.

ii. $\# V_m = (N_b - 1)\, N_b^m + 1$.

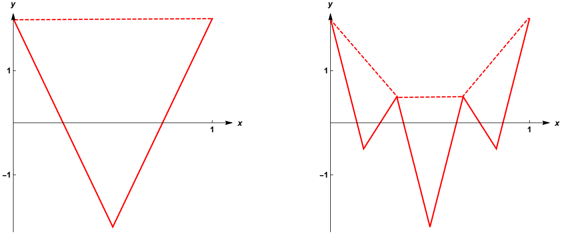


iii. The prefractal graph $\Gamma_{\mathscr{W}_m}$ has exactly $(N_b - 1)\, N_b^m$ edges.

iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathscr{W}_m}$ are the vertices of N_b^m **simple polygons** $\mathscr{P}_{m,k}$ with N_b sides. For $m \in \mathbb{N}$, the junction point between two consecutive polygons is the point

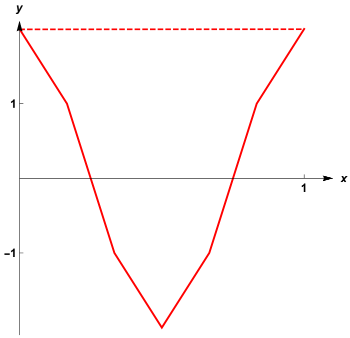
$$\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathscr{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m} \right) \right) \quad , \quad 1 \leq k \leq N_b^m - 1 \cdot$$

The total number of junction points is thus $N_b^m - 1$.
For instance, in the case $N_b = 3$, one gets triangles.
In the sequel, we will denote by $\mathscr{P}_0$ **the initial polygon**, i.e. the one whose vertices are the fixed points of the maps T_i, $0 \leq i \leq N_b - 1$.

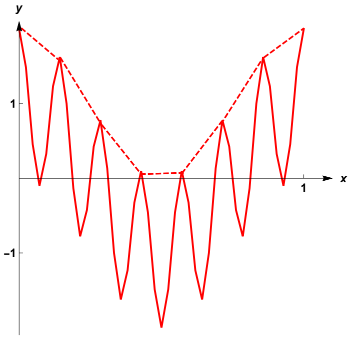


The polygons, in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.

The polygons, in the case where $\lambda = \frac{1}{2}$, and $N_b = 7$.

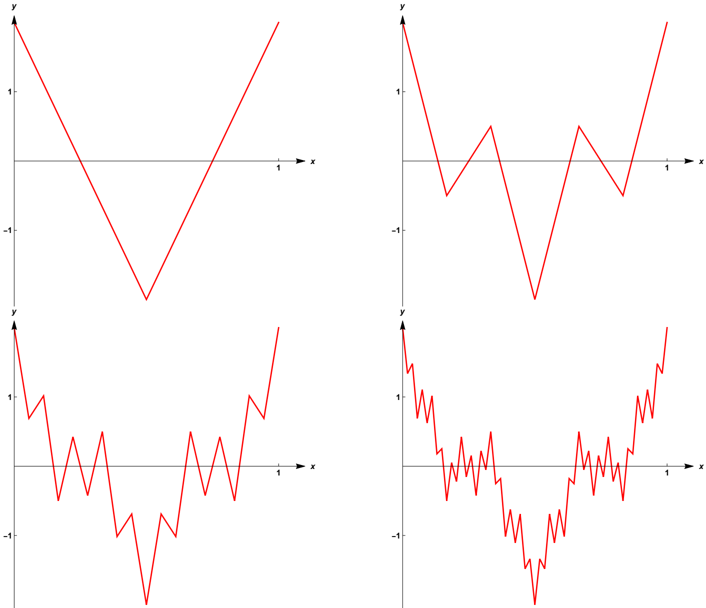


$m = 0$

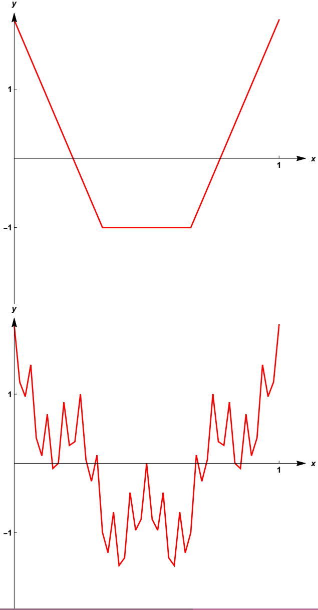


$m = 1$

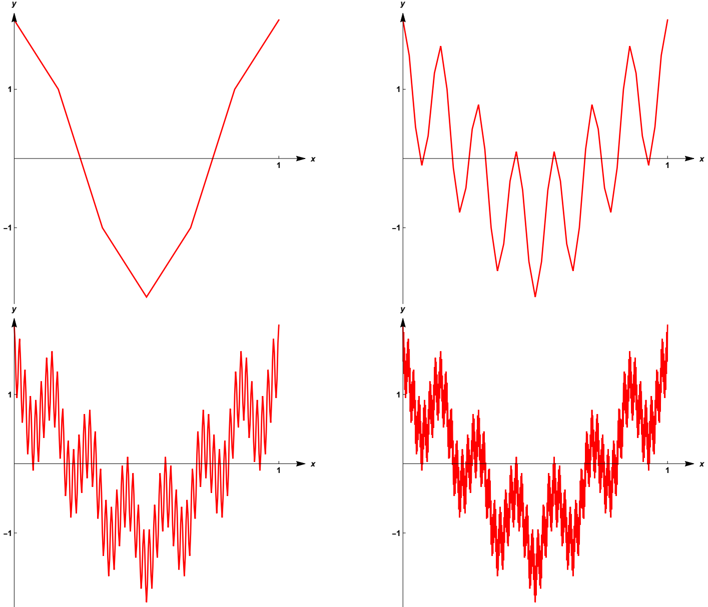
The prefractal graphs $\Gamma_{\mathscr{W}_0}$, $\Gamma_{\mathscr{W}_1}$, $\Gamma_{\mathscr{W}_2}$, $\Gamma_{\mathscr{W}_3}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 3$.



The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 4$.



The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, in the case where $\lambda = \frac{1}{2}$, and $N_b = 7$.



Vertices of the Prefractals, Elementary Lengths, and Heights

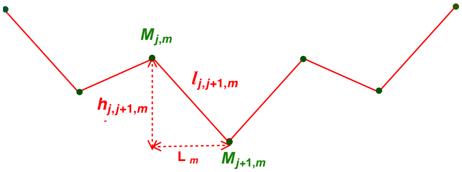
Given $m \in \mathbb{N}$, we denote by $\left(\boldsymbol{M_{j,m}}\right)_{0\leq j\leq (N_b-1)\,N_b^m-1}$ **the set of vertices** of the prefractal graph $\Gamma_{\mathscr{W}_m}$. One thus has, for any integer j in $\left\{0,\cdots,(N_b-1)\,N_b^m-1\right\}$:

$$M_{j,m}=\left(\frac{j}{(N_b-1)\,N_b^m},\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right).$$

We also introduce, for $0\leq j\leq (N_b-1)\,N_b^m-2$:

i. <u>the elementary horizontal lengths</u>:

$$L_m=\frac{1}{(N_b-1)\,N_b^m}$$

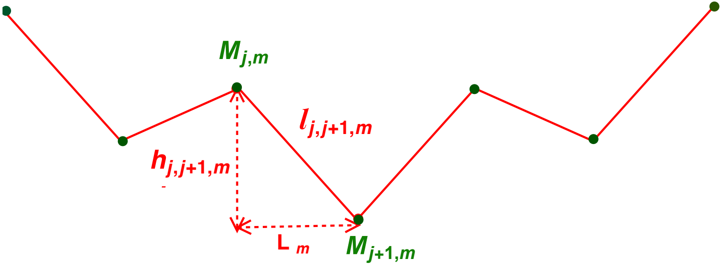


ii. the elementary lengths:

$$\ell_{j,j+1,m} = d\left(M_{j,m}, M_{j+1,m}\right) = \sqrt{L_m^2 + h_{j,j+1,m}^2}$$

iii. the elementary heights:

$$h_{j,j+1,m} = \left|\mathscr{W}\left(\frac{j+1}{\left(N_b-1\right)N_b^m}\right) - \mathscr{W}\left(\frac{j}{\left(N_b-1\right)N_b^m}\right)\right|$$



iv. the geometric angles:

$$\theta_{j-1,j,m} = \left((y'y), \left(\widehat{M_{j-1,m} M_{j,m}} \right) \right) \quad , \quad \theta_{j,j+1,m} = \left((y'y), \left(\widehat{M_{j,m} M_{j+1,m}} \right) \right),$$

which yield **the value of the geometric angle between consecutive edges** $\left[M_{j-1,m} \, M_{j,m}, M_{j,m} \, M_{j+1,m} \right]$:

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{\left| h_{j-1,j,m} \right|} + \arctan \frac{L_m}{\left| h_{j,j+1,m} \right|} \, .$$

Property (Scaling Properties of the Weierstrass Function, and Consequences)

Since, for any real number x

$$\mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\,N_b^n\,x\right)$$

one also has

$$\mathscr{W}(N_b\,x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\,N_b^{n+1}\,x\right) = \frac{1}{\lambda}\,\sum_{n=1}^{+\infty} \lambda^n \cos\left(2\,\pi\,N_b^n\,x\right) = \frac{1}{\lambda}\,\left\{\mathscr{W}(x) - \cos\left(2\,\pi\,x\right)\right\}$$

which yield, for any strictly positive integer m, and any j in $\{0,\cdots,\#V_m\}$*:*

$$\mathscr{W}\left(\frac{\boldsymbol{j}}{(\boldsymbol{N_b}-1)\,\boldsymbol{N_b^m}}\right) = \boldsymbol{\lambda}\,\mathscr{W}\left(\frac{\boldsymbol{j}}{(\boldsymbol{N_b}-1)\,\boldsymbol{N_b^{m-1}}}\right) + \cos\left(\frac{2\,\boldsymbol{\pi}\,\boldsymbol{j}}{(\boldsymbol{N_b}-1)\,\boldsymbol{N_b^{m-1}}}\right)$$

By induction, one obtains that

$$\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)=\lambda^m\,\mathscr{W}\left(\frac{j}{(N_b-1)}\right)+\sum_{k=0}^{m-1}\lambda^k\,\cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right).$$

A Consequence of the Symmetry with respect to the Vertical Line $x = \frac{1}{2}$

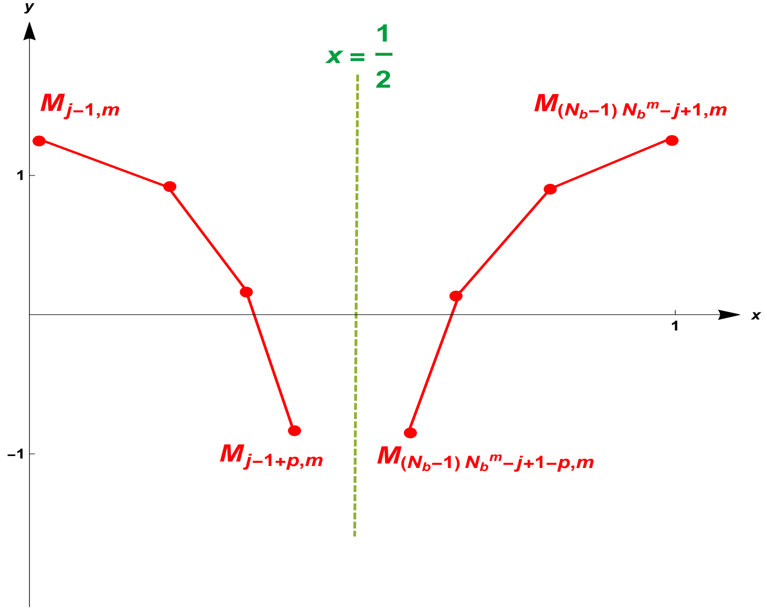
For any strictly positive integer m and any j in $\{0, \cdots, \# V_m\}$, we have that

$$\mathscr{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right) = \mathscr{W}\left(\frac{(N_b - 1)\, N_b^m - j}{(N_b - 1)\, N_b^m}\right)$$

which means that **the points**

$$\left(\frac{(N_b - 1)\, N_b^m - j}{(N_b - 1)\, N_b^m}, \mathscr{W}\left(\frac{(N_b - 1)\, N_b^m - j}{(N_b - 1)\, N_b^m}\right)\right) \quad \text{and} \quad \left(\frac{j}{(N_b - 1)\, N_b^m}, \mathscr{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right)\right)$$

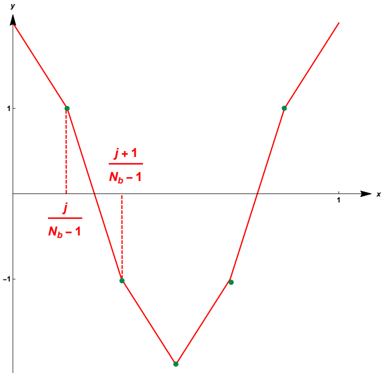
are symmetric with respect to the vertical line $x = \frac{1}{2}$.

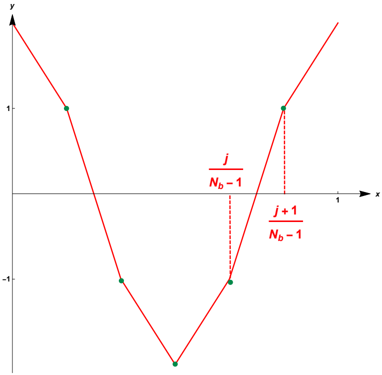

y
x
$x = \frac{1}{2}$
$M_{j-1,m}$
$M_{(N_b-1)\,N_b{}^m-j+1,m}$
1
−1
1
$M_{j-1+p,m}$
$M_{(N_b-1)\,N_b{}^m-j+1-p,m}$

Property

i. **For** $0 \leq j \leq \frac{(N_b - 1)}{2}$: $\mathscr{W}\left(\frac{j+1}{N_b-1}\right) - \mathscr{W}\left(\frac{j}{N_b-1}\right) \leq 0.$

ii. **For** $\frac{(N_b - 1)}{2} \leq j \leq N_b - 1$: $\mathscr{W}\left(\frac{j+1}{N_b-1}\right) - \mathscr{W}\left(\frac{j}{N_b-1}\right) \geq 0.$





Property

Given a strictly positive integer m:

i. For any j in $\{0, \cdots, \# V_m\}$, the point

$$\left(\frac{j}{(N_b-1)\, N_b^m}, \mathscr{W}\left(\frac{j}{(N_b-1)\, N_b^m} \right) \right)$$

is the image of the point

$$\left(\frac{j}{(N_b-1)\, N_b^{m-1}} - i, \mathscr{W}\left(\frac{j}{(N_b-1)\, N_b^{m-1}} - i \right) \right) = \left(\frac{j - i\,(N_b-1)\, N_b^{m-1}}{(N_b-1)\, N_b^{m-1}}, \mathscr{W}\left(\frac{j - i\,(N_b-1)\, N_b^{m-1}}{(N_b-1)\, N_b^{m-1}} \right) \right)$$

by the map T_i, $0 \leq i \leq N_b - 1$.

As a consequence, **the $\boldsymbol{j}^{th}$ vertex of the polygon** $\mathscr{P}_{m,k}$, $0 \leq k \leq N_b^m - 1$, $0 \leq j \leq N_b - 1$, i.e. the point:

$$\left(\frac{(N_b - 1)\, k + j}{(N_b - 1)\, N_b^m}, \mathscr{W}\left(\frac{(N_b - 1)\, k + j}{(N_b - 1)\, N_b^m} \right) \right)$$

is the image of the point

$$\left(\frac{(N_b - 1)\left(k - i\,(N_b - 1)\, N_b^{m-1}\right) + j}{(N_b - 1)\, N_b^{m-1}}, \mathscr{W}\left(\frac{(N_b - 1)\left(k - i\,(N_b - 1)\, N_b^{m-1}\right) + j}{(N_b - 1)\, N_b^{m-1}} \right) \right)$$

i.e. is the **the $\boldsymbol{j}^{th}$ vertex of the polygon** $\mathscr{P}_{m-1,k-i\,(N_b-1)\,N_b^{m-1}}$.
There is thus **an exact correspondence between vertices of the polygons at consecutive steps $\boldsymbol{m - 1}$, $\boldsymbol{m}$**.

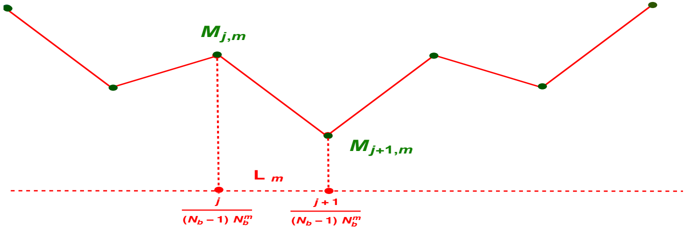
ii. Given j in $\{0, \cdots, N_b - 2\}$, and k in $\left\{0, \cdots, N_b^m - 1\right\}$:

$$\text{sign}\left(\mathscr{W}\left(\frac{k\,(N_b - 1) + j + 1}{(N_b - 1)\, N_b^m} \right) - \mathscr{W}\left(\frac{k\,(N_b - 1) + j}{(N_b - 1)\, N_b^m} \right) \right) = \text{sign}\left(\mathscr{W}\left(\frac{j + 1}{N_b - 1} \right) - \mathscr{W}\left(\frac{j}{N_b - 1} \right) \right).$$

Bounding Result: Upper and Lower Bounds for the Elementary Heights

For any strictly positive integer m, and any j in $\left\{0, \cdots, (N_b - 1)\, N_b^m\right\}$, we have that

$$C_{inf} \underbrace{\lambda^m}_{N_b^{m\,(D_{\mathscr{W}}-2)}} \leq \left| \mathscr{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right) - \mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) \right| \leq C_{sup} \underbrace{\lambda^m}_{N_b^{m\,(D_{\mathscr{W}}-2)}}$$



where

$$C_{inf} = (N_b - 1)^{2-D_{\mathscr{W}}} \min_{0 \le j \le N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathscr{W}}} \left(\max_{0 \le j \le N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right| + \frac{2\,\pi}{(N_b - 1)\,(\lambda\, N_b - 1)} \right).$$

These constants depend on the initial polygon $\mathscr{P}_0$.

Theorem: Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function

For any natural integer m, and any pair of real numbers (x, x') such that:

$$x = \frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m} = ((N_b-1)\,k+j)\;L_m \quad , \quad x' = \frac{(N_b-1)\,k+j+\ell}{(N_b-1)\,N_b^m} = ((N_b-1)\,k+j+\ell)\;L_m$$

where $0 \leq k \leq N_b - 1^m - 1$, and

i. if the integer N_b is odd,

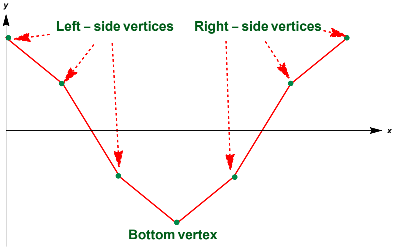
$$0 \leq j < \frac{N_b-1}{2} \quad \text{and} \quad 0 < j+\ell \leq \frac{N_b-1}{2}$$

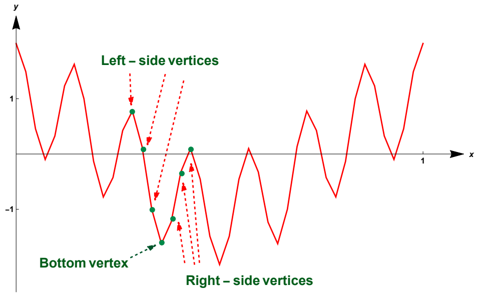
$$\text{or} \quad \frac{N_b-1}{2} \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b-1}{2} < j+\ell \leq N_b - 1\,;$$

ii. if the integer N_b is even,

$$0 \leq j < \frac{N_b}{2} \quad \text{and} \quad 0 < j+\ell \leq \frac{N_b}{2}$$

$$\text{or} \quad \frac{N_b}{2}+1 \leq j < N_b - 1 \quad \text{and} \quad \frac{N_b}{2}+1 < j+\ell \leq N_b - 1\,,$$





This means that the points $(x, \mathscr{W}(x))$ and $\left(x', \mathscr{W}(x')\right)$ are vertices of the polygon $\mathscr{P}_{m,k}$ **both located on the left-side of the polygon**, or **on the right-side**. Then, one has the following *reverse-Hölder inequality*, with **sharp Hölder exponent** $-\dfrac{\ln \lambda}{\ln N_b} = 2 - D_{\mathscr{W}}$,

$$C_{inf}\, \left|x' - x\right|^{2-D_{\mathscr{W}}} \leq \left|\mathscr{W}(x') - \mathscr{W}(x)\right| \cdot$$

Corollary

One may now write, for any $m \in \mathbb{N}^\star$, and $0 \leq j \leq (N_b - 1)\, N_b^m - 1$:

i. **for the elementary heights**:

$$h_{j-1,j,m} = L_m^{2-D_{\mathscr{W}}} \, \mathscr{O}(1)$$

ii. **for the elementary quotients**:

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathscr{W}}} \, \mathscr{O}(1)$$

where:

$$0 < C_{inf} \leq \mathscr{O}(1) \leq C_{sup} < \infty \cdot$$

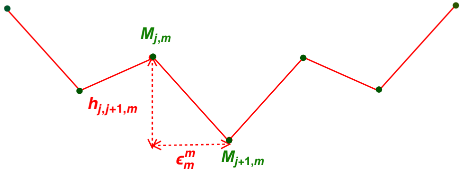
II. Polyhedral Measure

m^{th} Cohomology Infinitesimal

Given any $m \in \mathbb{N}$, we will call *m^{th} cohomology infinitesimal* the number

$$\varepsilon_m^m = \frac{1}{N_b - 1}\,\frac{1}{N_b^m} \underset{m\to\infty}{\to} 0 \cdot$$

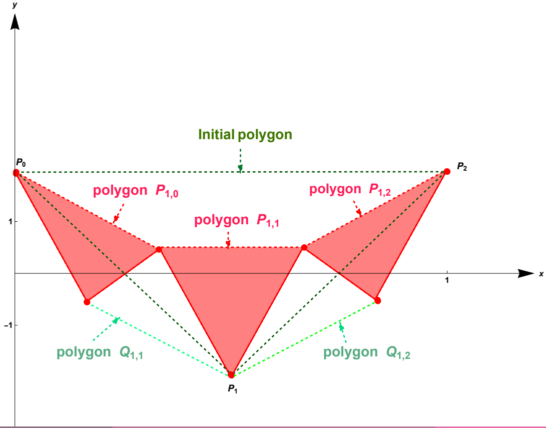
Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of $\mathscr{W}$.



Polygonal Sets

For any $m \in \mathbb{N}$, the consecutive vertices of the prefractal graph $\Gamma_{\mathscr{W}_m}$ are the vertices of N_b^m **simple polygons** $\mathscr{P}_{m,k}$ with N_b sides.
We now introduce the **polygonal sets**

$$\mathscr{P}_m = \left\{ \mathscr{P}_{m,k} \, , \, 0 \leq k \leq N_b^m - 1 \right\} \quad \text{and} \quad \mathscr{Q}_m = \left\{ \mathscr{Q}_{m,k} \, , \, 0 \leq k \leq N_b^m - 2 \right\} \cdot$$



Notation

For any $m \in \mathbb{N}$, we denote by:

ii. $X \in \mathscr{P}_m$ (resp., $X \in \mathscr{Q}_m$) a vertex of a polygon $\mathscr{P}_{m,k}$, with $0 \leq k \leq N_b^m - 1$ (resp., a vertex of a polygon $\mathscr{Q}_{m,k}$, with $1 \leq k \leq N_b^m - 2$).

ii. $\mathscr{P}_m \bigcup \mathscr{Q}_m$ the reunion of the polygonal sets $\mathscr{P}_m$ and $\mathscr{Q}_m$, which consists in the set of all the vertices of the polygons $\mathscr{P}_{m,k}$, with $0 \leq k \leq N_b^m - 1$, along with the vertices of the polygons $\mathscr{Q}_{m,k}$, with $1 \leq k \leq N_b^m - 2$. In particular, $X \in \mathscr{P}_m \bigcup \mathscr{Q}_m$ simply denotes a vertex in $\mathscr{P}_m$ or $\mathscr{Q}_m$.

iii. $\mathscr{P}_m \bigcap \mathscr{Q}_m$ the intersection of the polygonal sets $\mathscr{P}_m$ and $\mathscr{Q}_m$, which consists in the set of all the vertices of both a polygon $\mathscr{P}_{m,k}$, with $0 \leq k \leq N_b^m - 1$, and a polygon $\mathscr{Q}_{m,k'}$, with $1 \leq' k \leq N_b^m - 2$.

Power of a Vertex

Given $m \in \mathbb{N}^{\star}$, a vertex $\boldsymbol{X}$ of $\Gamma_{\mathscr{W}_m}$ is said:

i. *of* **power one** *relative to the polygonal family* $\mathscr{P}_m$ if X **belongs to** (or is a vertex of) **one and only one** N_b**-gon** $\mathscr{P}_{m,j}$, for $0 \leq j \leq N_b^m - 1$;

ii. *of* **power** $\frac{1}{2}$ *relative to the polygonal family* $\mathscr{P}_m$ if X is **a common vertex to two consecutive** N_b**-gons** $\mathscr{P}_{m,j}$ and $\mathscr{P}_{m,j+1}$, for $0 \leq j \leq N_b^m - 2$;

iii. *of* **power zero** *reative to the polygonal family* $\mathscr{P}_m$ if X **does not belong to** (or is not a vertex of) **any** N_b**-gon** $\mathscr{P}_{m,j}$, for $0 \leq j \leq N_b^m - 1$.

Similarly, given $m \in \mathbb{N}$, a vertex $\boldsymbol{X}$ of $\Gamma_{\mathscr{W}_m}$ is said:

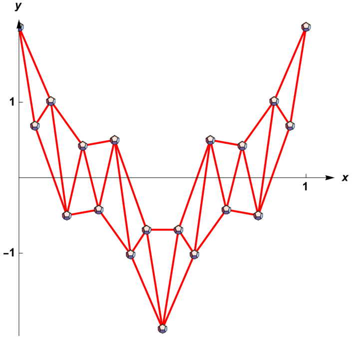
i. *of* **power one** *relative to the polygonal family* $\mathscr{Q}_m$ if X **belongs to** (or is a vertex of) **one and only one N_b-gon** $\mathscr{P}_{m,j}$, for $0 \leq j \leq N_b^m - 2$;

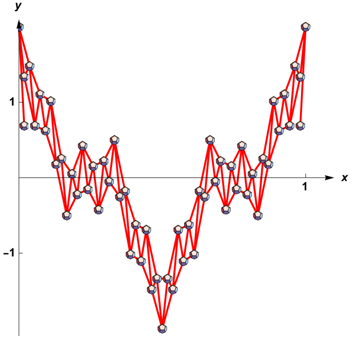
ii. *of* **power** $\frac{1}{2}$ *relative to the polygonal family* $\mathscr{P}_m$ if X is **a common vertex to two consecutive N_b-gons** $\mathscr{Q}_{m,j}$ and $\mathscr{Q}_{m,j+1}$, for $0 \leq j \leq N_b^m - 3$;

iii. *of* **power zero** *reative to the polygonal family* $\mathscr{P}_m$ if X **does not belong to** (or is not a vertex of) **any N_b-gon** $\mathscr{Q}_{m,j}$, for $0 \leq j \leq N_b^m - 2$.

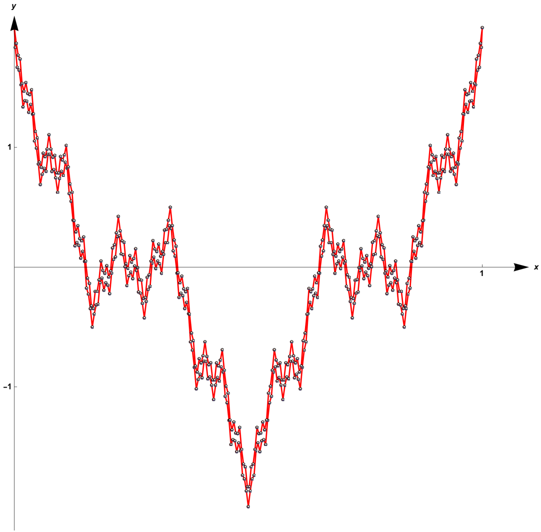
Sequence of Domains Delimited by the $\mathscr{W}$ IFD

We introduce *the sequence of domains delimited by the Weierstrass IFD* as the sequence $\left(\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $\left(\mathscr{P}_m \cup \mathscr{Q}_m\right)_{m\in\mathbb{N}}$, where, for each $m \in \mathbb{N}$, $\mathscr{P}_m$ and $\mathscr{Q}_m$ respectively denote the polygonal sets introduced just above.





$\mathscr{D}\left(\Gamma_{\mathscr{W}_2}\right)$ **and** $\mathscr{D}\left(\Gamma_{\mathscr{W}_3}\right)$**, for** $\lambda = \dfrac{1}{2}$ **and** $N_b = 3$**.**



$\mathscr{D}\left(\Gamma_{\mathscr{W}_5}\right)$, **for** $\boldsymbol{\lambda} = \frac{1}{2}$ **and** $\boldsymbol{N_b} = 3$**.**

Domain Delimited by the Weierstrass IFD

We call *domain, delimited by the Weierstrass IFD*, the set, which is equal to the following limit,

$$\mathscr{D}\left(\Gamma_{\mathscr{W}}\right) = \lim_{m\to\infty} \mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right),$$

where the convergence is interpreted **in the sense of the Hausdorff metric** on $\mathbb{R}^2$. In fact, we have that

$$\mathscr{D}\left(\Gamma_{\mathscr{W}}\right) = \Gamma_{\mathscr{W}} \cdot$$

Notation (Lebesgue Measure (on $\mathbb{R}^2$))

In the sequel, we denote by $\mu_{\mathscr{L}}$ the Lebesgue measure on $\mathbb{R}^2$.

Notation

For any $m \in \mathbb{N}$, and any vertex X of V_m, we set:

$$\mu^{\mathscr{L}}(X, \mathscr{P}_m, \mathscr{Q}_m) = \begin{cases} \dfrac{1}{N_b}\, p(X, \mathscr{P}_m) \displaystyle\sum_{0 \leq j \leq N_b^m - 1,\, X \text{ vertex of } \mathscr{P}_{m,j}} \mu_{\mathscr{L}}(\mathscr{P}_{m,j}) \,, \text{ if } X \notin \mathscr{Q}_m \,, \\ \dfrac{1}{N_b}\, p(X, \mathscr{Q}_m) \displaystyle\sum_{1 \leq j \leq N_b^m - 2,\, X \text{ vertex of } \mathscr{P}_{m,j}} \mu_{\mathscr{L}}(\mathscr{Q}_{m,j}) \,, \text{ if } X \notin \mathscr{P}_m \,, \\ \dfrac{1}{2\, N_b} \left\{ p(X, \mathscr{P}_m) \displaystyle\sum_{\substack{0 \leq j \leq N_b^m - 1, \\ X \text{ vertex of } \mathscr{P}_{m,j}}} \mu_{\mathscr{L}}(\mathscr{P}_{m,j}) + p(X, \mathscr{Q}_m) \displaystyle\sum_{\substack{1 \leq j \leq N_b^m - 2, \\ X \text{ vertex of } \mathscr{Q}_{m,j}}} \mu_{\mathscr{L}}(\mathscr{Q}_{m,j}) \right\} \\ \text{if } X \in \mathscr{P}_m \cap \mathscr{Q}_m \,. \end{cases}$$

Property

Given a continuous function u on $[0,1]\times[m_{\mathcal{W}},M_{\mathcal{W}}]$, we have that, for any $m\in\mathbb{N}$, and any vertex X of V_m:

$$\left|\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,u(X)\right|\leq\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\left(\max_{[0,1]\times[m_{\mathcal{W}},M_{\mathcal{W}}]}|u|\right)\lesssim N_b^{-(3-D_{\mathcal{W}})\,m}\,.$$

Consequently,we have that

$$\varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\left|\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,u(X)\right|\lesssim\varepsilon_m^{-m}\,\cdot$$

Since the sequence $\left(\displaystyle\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\varepsilon_m^{-m}\right)_{m\in\mathbb{N}}$ is **a positive and increasing sequence** (the number of vertices involved increases as m increases), this ensures the existence of the finite limit

$$\lim_{m\to\infty}\varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,u(X)\,\cdot$$

Theorem: Polyhedral Measure on the Weierstrass IFD ~ I

We introduce **the polyhedral measure** *on the Weierstrass IFD*, denoted by μ, such that for any continuous function u on the Weierstrass Curve,

$$\int_{\Gamma_{\mathscr{W}}} u \, d\mu = \lim_{m \to \infty} \varepsilon_m^{m(D_{\mathscr{W}}-2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}(X, \mathscr{P}_m, \mathscr{Q}_m)\, u(X)\,, \quad (\star)$$

which can also be understood in the following way,

$$\int_{\Gamma_{\mathscr{W}}} u \, d\mu = \int_{\mathscr{D}(\Gamma_{\mathscr{W}})} u \, d\mu \cdot$$

Theorem: Polyhedral Measure on the Weierstrass IFD ~ II

The polyhedral measure μ is **well defined, positive, as well as a bounded, nonzero, Borel measure** on $\mathscr{D}(\Gamma_{\mathscr{W}})$. The associated total mass is given by

$$\mu\left(\mathscr{D}\left(\Gamma_{\mathscr{W}}\right)\right) = \lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathscr{W}}-2)} \sum_{X\,\in\,\mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\left(X, \mathscr{P}_m, \mathscr{Q}_m\right)\,, \quad (\star\star)$$

and satisfies the following estimate:

$$\mu\left(\mathscr{D}\left(\Gamma_{\mathscr{W}}\right)\right) \leq \frac{2}{N_b}\left(N_b-1\right)^2 C_{sup} \cdot \quad (\star\star\star)$$

Furthermore, the support of μ coincides with the entire curve:

$$\operatorname{supp}\mu = \mathscr{D}\left(\Gamma_{\mathscr{W}}\right) = \Gamma_{\mathscr{W}} \cdot$$

Theorem - II

In addition, μ is the weak limit as $m \to \infty$ of *the following* **discrete measures** (or Dirac Combs), given, for each $m \in \mathbb{N}$, by

$$\mu_m = \varepsilon_m^{m\,(D_{\mathscr{W}} - 2)} \sum_{X \in \mathscr{P}_m \bigcup \mathscr{Q}_m} \mu^{\mathscr{L}}\,(X, \mathscr{P}_m, \mathscr{Q}_m)\;\delta_X\,,$$

where ε denotes the cohomology infinitesimal, and δ_X the Dirac measure concentrated at X.

III. Polyhedral and Tubular Neighborhoods

Classical Approach

Aim: Following [XVII], [XVIII] and [XIX]

↓

one requires fractal tube formulae for the IFD

i.e., the area of a two-sided ϵ-neighborhood of each prefractal approximation.

[XVII] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

[XVIII] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. "Fractal tube formulas for compact sets and relative fractal drums: Oscillations, complex dimensions and fractality". In: *Journal of Fractal Geometry. Mathematics of Fractals and Related Topics* 5.1 (2018), pp. 1–119.

[XIX] Michel L. Lapidus. "An overview of complex fractal dimensions: From fractal strings to fractal drums, and back". In: *Horizons of Fractal Geometry and Complex Dimensions*. Vol. 731. Contemporary Mathematics. Amer. Math. Soc., Providence, RI, 2019, pp. 143–265.

The fractal tube formula is expected to consist of an expansion of the form,

in the case of simple Complex Dimensions,

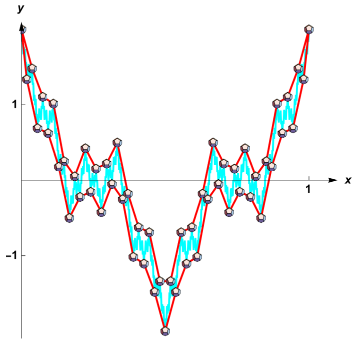
$$\sum_{\alpha \text{ real part of a Complex Dimension}} \epsilon^{2-\alpha}\, G_\alpha\left(\ln_{N_b}\left(\frac{1}{\epsilon}\right)\right), \qquad (\star)$$

(apart from ponctual terms) where, for any real part α of a Complex Dimension, G_α denotes **a continuous and one-periodic function**.

Instead of Tubular Neighborhoods

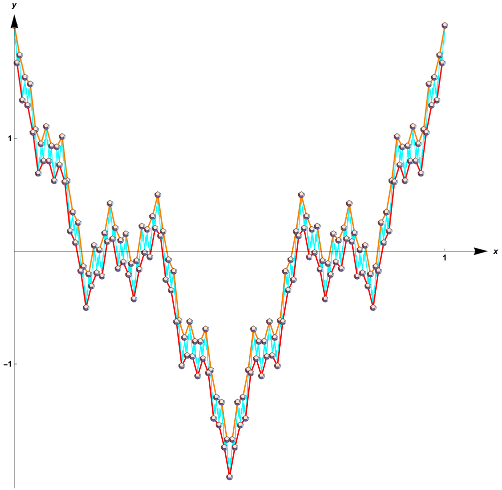
We can also consider

Polyhedral Neighborhoods



Polyhedral Neighborhood

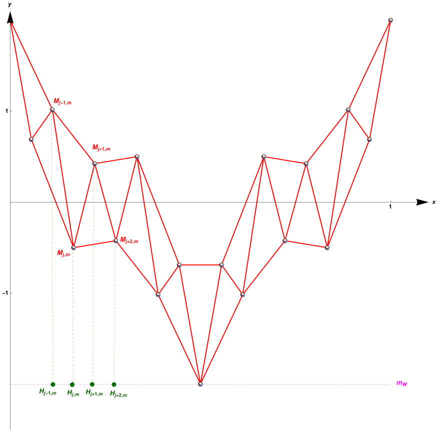
We consider *the sequence of domains delimited by the Weierstrass IFD* as the sequence $\left(\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $\left(\mathscr{P}_m \cup \mathscr{Q}_m\right)_{m\in\mathbb{N}}$.
Given $\in\mathbb{N}$, $\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)$ is the m^{th} *polyhedral neighborhood* (of the Weierstrass Curve).



Exact Expression

In the case where $N_b = 3$, given $m \in \mathbb{N}^\star$, the volume (or two-dimensional Lebesgue measure) of the m^{th}-polygonal neighborhood $\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)$ is given by

$$\mathscr{V}_m(\varepsilon_m^m) = \mu_{\mathscr{L}}\left(\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)\right) = \frac{\varepsilon_m^m}{2}\left(\mathscr{W}(0) + \mathscr{W}\left(2\,\varepsilon_m^m\right) - 2\,\mathscr{W}\left(\varepsilon_m^m\right)\right) \cdot$$



Proof

i. For $1 \leq j \leq \# V_m - 2$,

$$\mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right) = \frac{\varepsilon_m^m}{2}\left(2\,\mathscr{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right) - \mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) - \mathscr{W}\left(\frac{j+2}{(N_b-1)\,N_b^m}\right)\right).$$

ii. For $1 \leq j \leq \# V_m - 1$,

$$\mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right) = \frac{\varepsilon_m^m}{2}\left(\mathscr{W}\left(\frac{j-1}{(N_b-1)\,N_b^m}\right) + \mathscr{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right) - 2\,\mathscr{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right).$$

iii. We then have that

$$\mathscr{V}_m(\varepsilon_m^m) = \sum_{j=1}^{\# V_m - 3}\left(\mu_{\mathscr{L}}\left(\mathscr{P}_{m,j}\right) + \mu_{\mathscr{L}}\left(\mathscr{Q}_{m,j}\right)\right) + \mu_{\mathscr{L}}\left(\mathscr{P}_{m,N_b^m}\right) = \frac{\varepsilon_m^m}{2}\left(1 + \mathscr{W}\left(2\,\varepsilon_m^m\right) - 2\,\mathscr{W}\left(\varepsilon_m^m\right)\right),$$

since, thanks **to the symmetry** with respect to the vertical line $x = \frac{1}{2}$,

$$\mathscr{W}\left(\frac{(N_b-1)\,N_b^m - 1}{(N_b-1)\,N_b^m}\right) = \mathscr{W}\left(\frac{1}{(N_b-1)\,N_b^m}\right) \quad \text{and} \quad \mathscr{W}\left(\frac{(N_b-1)\,N_b^m - 2}{(N_b-1)\,N_b^m}\right) = \mathscr{W}\left(\frac{2}{(N_b-1)\,N_b^m}\right).$$

Comparison with Tubular Neighborhoods

In the sequel, we denote by d the Euclidean distance.

Given a natural integer m, we introduce:

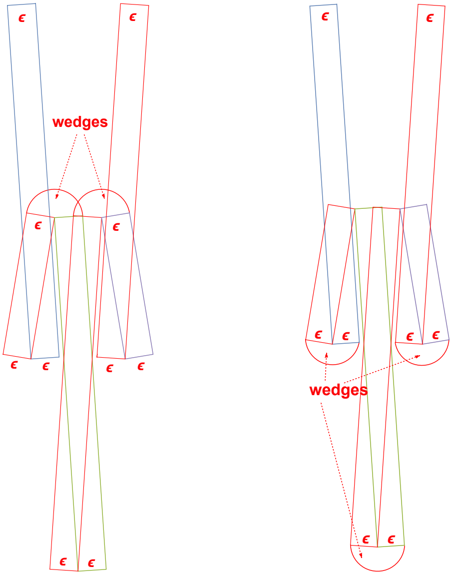
i. **The (m, ε_m^m)-Upper Neighborhood**:

$$\mathscr{D}^+\left(\Gamma_{\mathscr{W}_m}, \varepsilon_m^m\right) = \left\{M = (x, y) \in \mathbb{R}^2, y \geq \mathscr{W}(x) \text{ and } d\left(M, \Gamma_{\mathscr{W}_m}\right) \leq \varepsilon_m^m\right\} \cdot$$

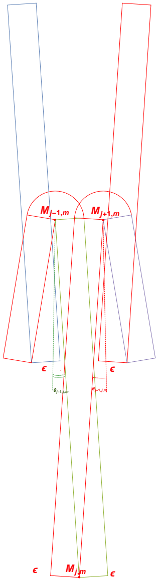
ii. **The (m, ε_m^m)-Lower Neighborhood**:

$$\mathscr{D}^-\left(\Gamma_{\mathscr{W}_m}, \varepsilon_m^m\right) = \left\{M = (x, y) \in \mathbb{R}^2, y \leq \mathscr{W}(x) \text{ and } d\left(M, \Gamma_{\mathscr{W}_m}\right) \leq \varepsilon_m^m\right\} \cdot$$

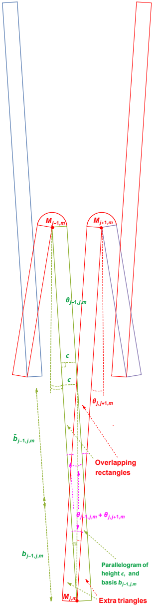
The (m, ε_m^m)-upper and lower Neighborhoods are then obtained by means of **rectangles** and **wedges**.



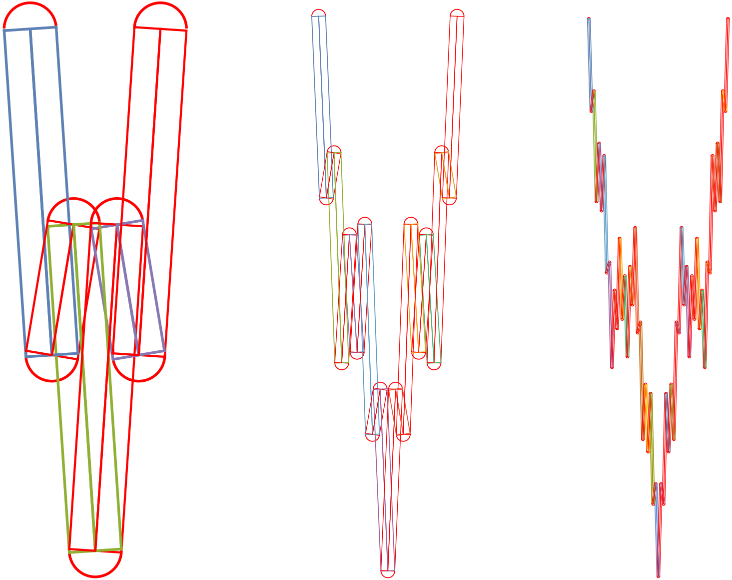
The $(1, \varepsilon_1^1)$-Upper Neighborhood, in the case where $\lambda = \dfrac{1}{2}$ and $N_b = 3$.



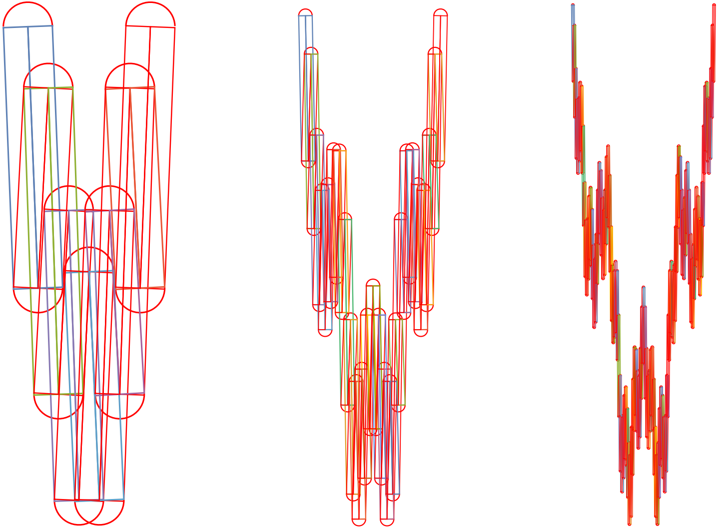
Two overlapping rectangles, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.



The $(1, \varepsilon_1^1)$, $(2, \varepsilon_2^2)$ and $(3, \varepsilon_3^3)$-Neighborhoods, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.



The $(1, \varepsilon_1^1)$, $(2, \varepsilon_2^2)$ and $(3, \varepsilon_3^3)$-Neighborhoods, in the case where $\lambda = \frac{1}{2}$ and $N_b = 4$.



Proposition: (m,ε_m^m)-Upper Neighborhood

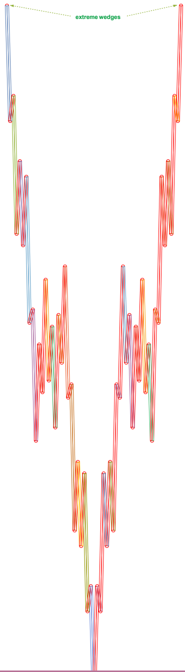
Given a strictly positive integer m, the (m,ε_m^m)-**Upper Neighborhood** is constituted of:

i. $(N_b-1)\,N_b^m$ **overlapping rectangles**, each of length $\ell_{j-1,j,m}$, $1\leq j\leq N_b^m-1$, and height ε_m^m.
The area that is counted twice corresponds to parallelograms, of height ε_m^m and basis $\varepsilon_m^m \operatorname{cotan}\left(\pi-\theta_{j-1,j,m}-\theta_{j,j+1,m}\right)$.
Since one deals here with an upper neighborhood, one also has to substract the areas of the **extra outer lower triangles**.

ii. $N_b^m\left(1+2\left[\dfrac{N_b-3}{4}\right]\right)-1$ **upper wedges**. The number of wedges is determined by the shape of the initial polygon $\mathscr{P}_0$, as well by the existence of reentrant angles.

iii. Two **extreme wedges**, each of area

$$\frac{1}{2}\,\pi\,\left(\varepsilon_m^m\right)^2\,.$$


extreme wedges

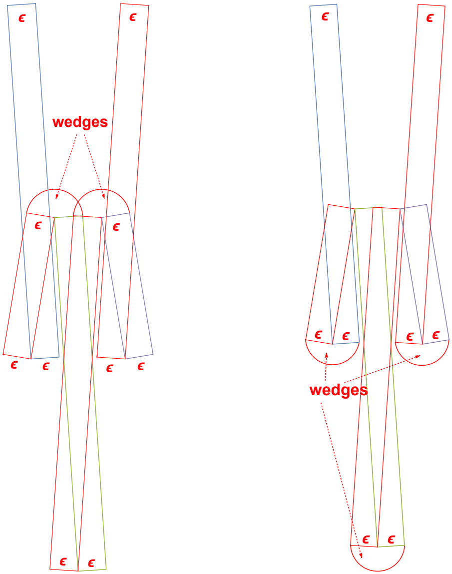
Proposition: (m, ε_m^m)-Lower Neighborhood

In the same way, given a strictly positive integer m, the (m, ε_m^m)-**Lower Neighborhood** is thus constituted of:

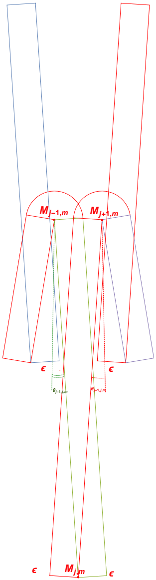
i. $(N_b - 1)\, N_b^m$ **overlapping rectangles**, each of length $\ell_{j-1,j,m}$, $1 \leq j \leq N_b^m - 1$, and height ε_m^m.
The area that is thus counted twice again corresponds to parallelograms, of height ε_m^m and basis $\varepsilon_m^m \operatorname{cotan}\left(\pi - \theta_{j-1,j,m} - \theta_{j,j+1,m}\right)$.
Since one deals here with a lower neighborhood, one has this time to substract the areas of the upper **extra outer upper triangles**.

ii. $N_b^m \left(N_b - 2 \left[\dfrac{N_b - 3}{4} \right] \right) - 1$ **lower wedges**.
The number of lower wedges is determined by the shape of the initial polygon $\mathscr{P}_0$, as well as by the existence of reentrant angles.

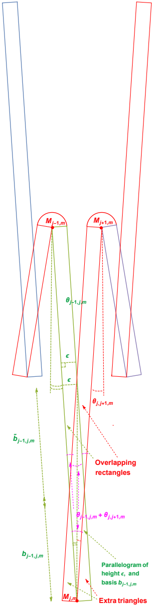
The (m, ε_m^m)-upper and lower Neighborhoods are then obtained by means of **rectangles** and **wedges**.



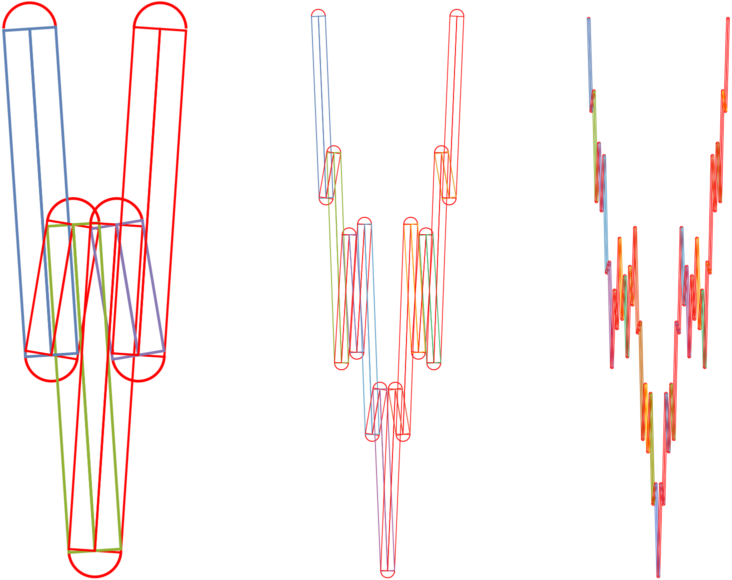
The $(1, \varepsilon_1^1)$-Upper Neighborhood, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.



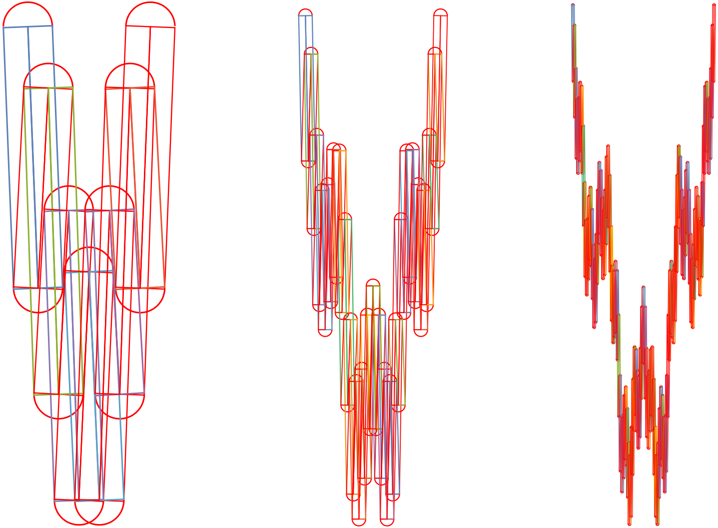
Two overlapping rectangles, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.



The $(1, \varepsilon_1^1)$, $(2, \varepsilon_2^2)$ and $(3, \varepsilon_3^3)$-Neighborhoods, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$.



The $(1,\varepsilon_1^1)$, $(2,\varepsilon_2^2)$ and $(3,\varepsilon_3^3)$-Neighborhoods, in the case where $\lambda = \frac{1}{2}$ and $N_b = 4$.



Proposition: (m, ε_m^m)-Upper Neighborhood

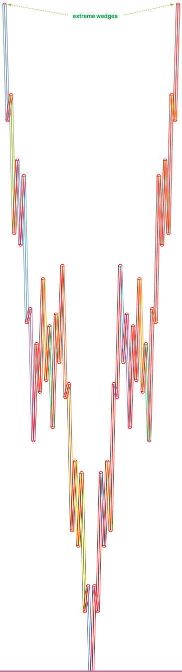
Given a strictly positive integer m, the (m, ε_m^m)-**Upper Neighborhood** is constituted of:

i. $(N_b - 1)\, N_b^m$ **overlapping rectangles**, each of length $\ell_{j-1,j,m}$, $1 \leq j \leq N_b^m - 1$, and height ε_m^m.
The area that is counted twice corresponds to parallelograms, of height ε_m^m and basis $\varepsilon_m^m \operatorname{cotan}\left(\pi - \theta_{j-1,j,m} - \theta_{j,j+1,m}\right)$.
Since one deals here with an upper neighborhood, one also has to substract the areas of the **extra outer lower triangles**.

ii. $N_b^m \left(1 + 2\left[\dfrac{N_b - 3}{4}\right]\right) - 1$ **upper wedges**. The number of wedges is determined by the shape of the initial polygon $\mathscr{P}_0$, as well by the existence of reentrant angles.

iii. Two **extreme wedges**, each of area

$$\frac{1}{2}\,\pi\,\left(\varepsilon_m^m\right)^2 \,.$$


extreme wedges

Proposition: (m, ε_m^m)-Lower Neighborhood

In the same way, given a strictly positive integer m, the (m, ε_m^m)-**Lower Neighborhood** is thus constituted of:

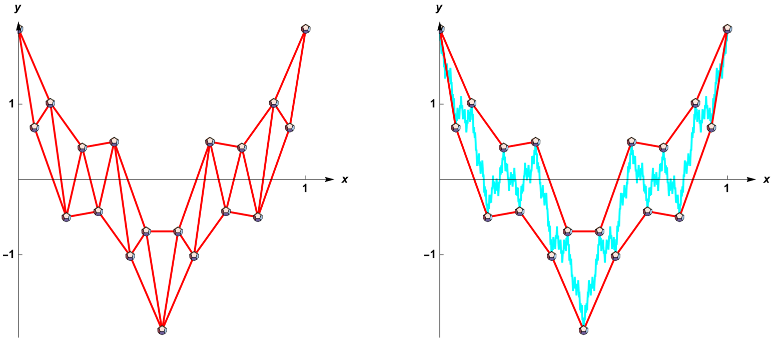
i. $(N_b - 1)\, N_b^m$ **overlapping rectangles**, each of length $\ell_{j-1,j,m}$, $1 \leq j \leq N_b^m - 1$, and height ε_m^m.
The area that is thus counted twice again corresponds to parallelograms, of height ε_m^m and basis $\varepsilon_m^m \operatorname{cotan}\left(\pi - \theta_{j-1,j,m} - \theta_{j,j+1,m}\right)$.
Since one deals here with a lower neighborhood, one has this time to substract the areas of the upper **extra outer upper triangles**.

ii. $N_b^m \left(N_b - 2 \left[\dfrac{N_b - 3}{4} \right] \right) - 1$ **lower wedges**.
The number of lower wedges is determined by the shape of the initial polygon $\mathscr{P}_0$, as well as by the existence of reentrant angles.

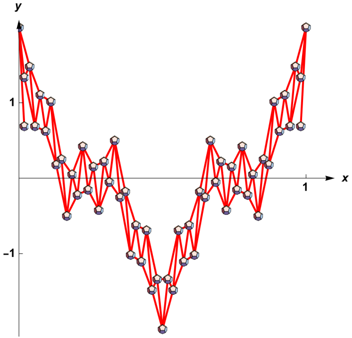
Theorem: The Nested Neighborhoods

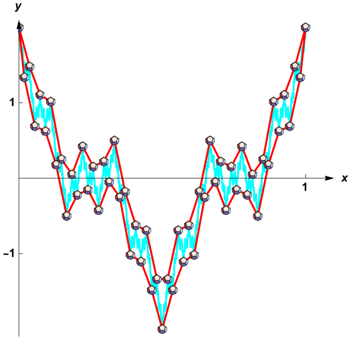
i. Given $m \in \mathbb{N}$, there exists $m_1 \in \mathbb{N}$ such that, for all $k \geq m_1$, **the polyhedral neighborhood** $\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)$ **contains, but for a finite number of wedges**, **the** $(m+k, \varepsilon_{m+k}^{m+k})$ **tubular neighborhood** $\mathscr{D}^{\boldsymbol{tube}}\left(\Gamma_{\mathscr{W}_{m+k}}, \boldsymbol{\varepsilon_{m+k}^{m+k}}\right)$.

ii. Given $m \in \mathbb{N}$, there exists $m_2 \in \mathbb{N}$ such that, for all $k \geq m_2$, **the tubular** (m, ε_m^m)**-neighborhood** $\mathscr{D}^{\boldsymbol{tube}}\left(\Gamma_{\mathscr{W}_m}, \boldsymbol{\varepsilon_m^m}\right)$ **contains the polyhedral neighborhood** $\mathscr{D}\left(\Gamma_{\mathscr{W}_{m+k}}\right)$.



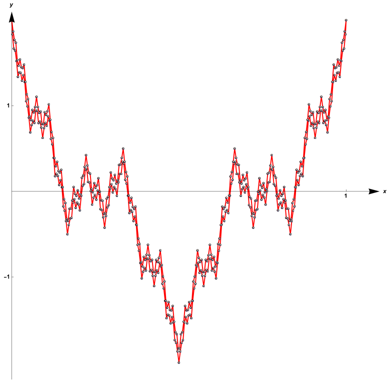
$\mathscr{D}\left(\Gamma_{\mathscr{W}_2}\right)$ **(in red), and** $\mathscr{D}^{tube}\left(\Gamma_{\mathscr{W}_7}, \varepsilon_7^7\right)$**, in the case where** $\lambda = \dfrac{1}{2}$ **and** $N_b = 3$**.**

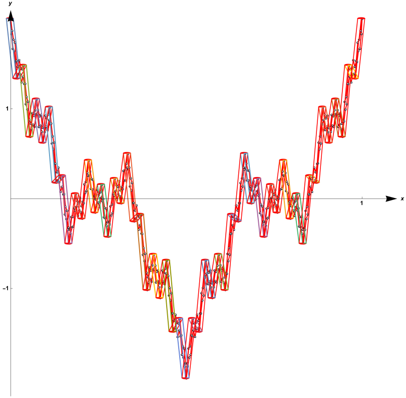




The polygonal neigborhood $\mathscr{D}\left(\Gamma_{\mathscr{W}_3}\right)$, in the case where $\lambda = \dfrac{1}{2}$ and $N_b = 3$.

$\mathscr{D}\left(\Gamma_{\mathscr{W}_3}\right)$ **(in red), and** $\mathscr{D}^{tube}\left(\Gamma_{\mathscr{W}_7}, \varepsilon_7^7\right)$**, in the case where** $\lambda = \dfrac{1}{2}$ **and** $N_b = 3$**.**





$\mathscr{D}\left(\Gamma_{\mathscr{W}_5}\right)$ **and** $\mathscr{D}^{tube}\left(\Gamma_{\mathscr{W}_3}, \varepsilon_3^3\right)$**, in the case where** $\lambda = \frac{1}{2}$ **and** $N_b = 3$**.**

Proof

i. At a given step $m \geq 0$, between two adjacent vertices $M_{i,m}$ and $M_{i+1,m}$ of V_m, there are $N_b - 1$ consecutive vertices of $V_{m+1} \setminus V_m$, $\left(M_{j+1,m+1}, \cdots, M_{j+N_b-2,m+1}\right) \in V$ such that

$$M_{i,m} = M_{j,m+1} \quad \text{and} \quad M_{i+1,m} = M_{j+N_b,m+1} \, .$$

We dispose of an exact correspondance between vertices of the polygons at the step $m+1$, and at the initial step $m = 0$. Since reentrant angles occur when $N_b \geq 7$, we can restrict ourselves to the cases $N_b \leq 6$ (in the case of reentrant angles, the following arguments can be suitably adjusted). We then simply have to consider the $\left[\dfrac{N_b - 2}{2}\right]$ vertices $M_{j+k,m+1}$, with $1 \leq k \leq \left[\dfrac{N_b - 2}{2}\right]$ (the same arguments holds for the vertices $M_{j+N_b-k,m+1}$). Given j in $\{0, \cdots, N_b - 2\}$ and k in $\left\{0, \cdots, N_b^{m+1} - 1\right\}$, we have that

$$\operatorname{sgn}\left(\mathscr{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^{m+1}}\right) - \mathscr{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^{m+1}}\right)\right) = \operatorname{sgn}\left(\mathscr{W}\left(\frac{j+1}{N_b-1}\right) - \mathscr{W}\left(\frac{j}{N_b-1}\right)\right),$$

i.e., equivalently,

$$\operatorname{sgn}\left(\mathscr{W}\left(\left(k\left(N_b-1\right)+j+1\right)L_{m+1}\right)-\mathscr{W}\left(\left(k\left(N_b-1\right)+j\right)L_{m+1}\right)\right)=\operatorname{sgn}\left(\mathscr{W}\left(\frac{j+1}{N_b-1}\right)-\mathscr{W}\left(\frac{j}{N_b-1}\right)\right).$$

Due to the symmetry of the initial polygon $\mathscr{P}_0$ (or, equivalently, of the initial prefractal graph $\Gamma_{\mathscr{W}_0}$) with respect to the vertical line $x=\dfrac{1}{2}$ (see Property 1), this means that we can restrict ourselves to the case when

$$\mathscr{W}\left(j\,L_{m+1}\right)\geq\mathscr{W}\left(\left(j+1\right)L_{m+1}\right)\geq\cdots\geq\mathscr{W}\left(\left(j+\left[\frac{N_b-2}{2}\right]\right)L_{m+1}\right),$$

and

$$\underbrace{\mathscr{W}\left(j\,L_{m+1}\right)}_{\mathscr{W}\left(i\,L_m\right)}\geq\underbrace{\mathscr{W}\left(\left(j+N_b\right)\,L_{m+1}\right)}_{\mathscr{W}\left(\left(i+1\right)L_m\right)},$$

since

$$M_{i,m}=M_{j,m+1}\quad\text{and}\quad M_{i+1,m}=M_{j+N_b,m+1}\,.$$

We then deduce, by triangle inequality, for $1 \leq k \leq \left[\dfrac{N_b-2}{2}\right]$, that

$$\left| \mathscr{W}\left((j+k)\,L_{m+1}\right) - \underbrace{\mathscr{W}\left(j\,L_{m+1}\right)}_{\mathscr{W}\left(i\,L_m\right)} \right| \leq \left[\frac{N_b-2}{2}\right] C_{sup}\, L_{m+1}^{2-D_{\mathscr{W}}} \,.$$

Since

$$L_{m+1} = \frac{L_m}{N_b}\,,$$

we then obtain that

$$\left| \mathscr{W}\left((j+k)\,L_{m+1}\right) - \underbrace{\mathscr{W}\left(j\,L_{m+1}\right)}_{\mathscr{W}\left(i\,L_m\right)} \right| \leq \left[\frac{N_b-2}{2}\right] N_b^{D_{\mathscr{W}}-2}\, C_{sup}\, L_m^{2-D_{\mathscr{W}}} \,.$$

Recall now that

$$C_{inf} = (N_b - 1)^{2-D_{\mathscr{W}}} \min_{0 \leq j \leq N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right| ,$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathscr{W}}} \left(\max_{0 \leq j \leq N_b - 1} \left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right| + \frac{2\,\pi}{(N_b - 1)\,(\lambda\, N_b - 1)} \right) \cdot$$

Here, we have that

$$\mathscr{W}\left(\frac{j}{N_b - 1}\right) = \frac{1}{1 - \lambda} \cos \frac{2\,\pi\, j}{N_b - 1} \cdot$$

This ensures that

$$\left| \mathscr{W}\left(\frac{j+1}{N_b - 1}\right) - \mathscr{W}\left(\frac{j}{N_b - 1}\right) \right| \leq \frac{2\,\pi}{N_b - 1}\, \frac{1}{1 - \lambda} \cdot$$

We can check numerically that

$$\left[\frac{N_b-2}{2}\right] N_b^{D_{\mathscr{W}}-2}\, C_{sup} \leq C_{inf}\,,$$

from which we immediately deduce that for, $1 \leq k \leq \left[\dfrac{N_b-2}{2}\right]$,

$$\left| \mathscr{W}\left((j+k)\,L_{m+1}\right) - \underbrace{\mathscr{W}\left(j\,L_{m+1}\right)}_{\mathscr{W}\left(i\,L_m\right)} \right| \leq \left| \mathscr{W}\left((i+1)\,L_m\right) - \mathscr{W}\left(i\,L_m\right) \right| \cdot$$

For $1 \leq k \leq \left[\dfrac{N_b - 2}{2}\right]$, the vertices $M_{j+k,m+1}$ are then strictly between the vertices $M_{i,m}$ and $M_{i+1,m}$. As is explained previously, we can show, in a similar way, that for $1 \leq k \leq \left[\dfrac{N_b - 2}{2}\right]$, the vertices $M_{j+N_b-k,m+1}$ are also strictly between the vertices $M_{i,m}$ and $M_{i+1,m}$.

By induction, we then obtain that, given four consecutive adjacent vertices $M_{i,m}$, $M_{i+1,m}$, $M_{i+3,m}$ and $M_{i+4,m}$ of V_m, with $1 \leq i \leq \# V_m - 5$ and $k \in \mathbb{N}$, the vertices of $V_{m+k} \setminus V_m$ located between $M_{i,m}$ and $M_{i+4,m}$ can be all comprised in the simple and convex polygon $M_{i,m}M_{i+1,m}M_{i+3,m}M_{i+4,m}$, which coincides with the union of two consecutive polygons $\mathscr{P}_{m,j}$ and $\mathscr{Q}_{m,j}$. Thus, there exists $m_0 \in \mathbb{N}$ such that, for all $k \geq m_0$, the $(m+k, \varepsilon_{m+k}^{m+k})$-neighborhood

$$\mathscr{D}\left(\Gamma_{\mathscr{W}_{m+k}}, \varepsilon_{m+k}^{m+k}\right) = \left\{M = (x,y) \in \mathbb{R}^2 \,,\, d\left(M, \Gamma_{\mathscr{W}_{m+k}}\right) \leq \varepsilon_{m+k}^{m+k}\right\} \,,$$

from which we remove the wedges associated to the vertices $M_{i,m}$, $M_{i+1,m}$, $M_{i+3,m}$ and $M_{i+4,m}$ (see[XX]), can be totally included in the polygon $M_{i,m}M_{i+1,m}M_{i+3,m}M_{i+4,m}$. Hence, there exists $m_1 \in \mathbb{N}$ such that, for all $k \geq m_1$, the (m, ε_m^m)-neighborhood but for a finite number of wedges, the $(m+k, \varepsilon_{m+k}^{m+k})$-neighborhood $\mathscr{D}\left(\Gamma_{\mathscr{W}_{m+k}}, \varepsilon_{m+k}^{m+k}\right)$, can be totally included in the polygonal domain $\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)$.

[XX] Claire David and Michel L. Lapidus. *Weierstrass fractal drums - I - A glimpse of complex dimensions*. 2022.

ii. This latter result has been obtained in[XXI]. It comes from the fact that, in the sense of the Hausdorff metric on $\mathbb{R}^2$,

$$\lim_{m\to\infty} \mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right) = \Gamma_{\mathscr{W}} \cdot$$

[XXI] Claire David and Michel L. Lapidus. *Iterated fractal drums ~ Some New Perspectives: Polyhedral Measures, Atomic Decompositions and Morse Theory*. 2022.

IV. Zeta Functions

$\sim$

Complex Dimensions

Zeta functions ?

They represent the trace of a differential operator at a complex order s

$$\downarrow$$

Poles: Maximal Orders of Differentiation

$$\downarrow$$

Dimensions

Difficulty:

In our present context, when it comes to obtain **the associated fractal tube zeta function**, we cannot, as in the case of an arbitrary subset of $\mathbb{R}^2$ (see[XXII], Def. 2.2.8, p. 118), directly use an integral formula of the form

$$\tilde{\zeta}_m(s) = \int_0^{\varepsilon_m^m} t^{s-2}\, \mathscr{V}_m(t)\, \frac{dt}{t}\,,$$

since **the tube formulas can only be expressed in an explicit way at a cohomology infinitesimal**.

However, we can use **Riemann sums**, for the following nonuniform partition of the interval $\left[0, \varepsilon_m^m\right]$, where $k \to \infty$,

$$\left[0, \varepsilon_m^m\right] = \left[0, \varepsilon_{m\,k}^{m\,k}\right] \bigcup \left\{ \bigcup_{m+k+p=m\,k-2}^{m+k+p=m+k} \left[\varepsilon_{m+k+p+1}^{m+k+p+1}, \varepsilon_{m+k+p}^{m+k+p}\right] \right\} \bigcup \left[\varepsilon_{m+k}^{m+k}, \varepsilon_m^m\right].$$

[XXII] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

Theorem: m^{th}-Prefractal Effective Polyhedral Zeta Function ~ I

Given $m \in \mathbb{N}$, we introduce the m^{th}**-prefractal effective polyhedral zeta function** $\tilde{\zeta}$ such that, for admissible values of the complex number s,

$$\tilde{\zeta}^e_m(s) = \int_0^{\varepsilon_m^m} t^{s-3}\, \widetilde{\mathscr{V}}_m(t)\, dt = \frac{\varepsilon_m^{m\,(s-1)}}{2\,(s-1)} + \int_0^{\varepsilon_m^m} t^{s-3}\, \left(\mathscr{W}\,(2\,t) - 2\,\mathscr{W}\,(t)\right)\, dt\,,$$

where $\widetilde{\mathscr{V}}_m$ is *the volume extension function* associated with $\mathscr{V}_m$.

The associated sequence $\left(\tilde{\zeta}^e_m\right)_{m\in\mathbb{N}}$ satisfies the following recurrence relation, for values of the integer m sufficiently large,

$$\tilde{\zeta}^e_{m+1}(s) \quad = \quad N_b^{3-s}\, \tilde{\zeta}^e_m(s) + \frac{1}{2}\,(1-\lambda)\, N_b^{3-s}\, \frac{\varepsilon_m^{m\,(s-1)}}{s-1} + N_b^{3-s} \int_0^{\varepsilon_m^m} t^{s-2}\, \frac{1}{2\,N_b}\, \mathscr{R}e \left(e^{i\,4\,\pi\,t} - 2\,e^{i\,2\,\pi\,t}\right) dt\,.$$

Theorem: m^{th}-Prefractal Effective Polyhedral Zeta Function ~ II

This ensures the existence of the limit fractal zeta function, i.e., the fractal zeta function associated with the Weierstrass Curve $\Gamma_{\mathscr{W}}$, given by

$$\tilde{\zeta}^{e}_{\mathscr{W}} = \lim_{m \to \infty} \tilde{\zeta}^{e}_{m} \, ,$$

along with the existence of an integer $m_0 \in \mathbb{N}$ such that the poles of $\tilde{\zeta}_{\mathscr{W}}$ are the same as the poles of the fractal zeta function $\tilde{\zeta}^{e}_{m_0}$.

Proof

i. For sufficienly large values of $m \in \mathbb{N}$, i.e., $m \geq m_0$, for some suitable integer $m_0 \in \mathbb{N}$,

$$\tilde{\zeta}^e_{m+1}(s) \quad = \quad \int_0^{\varepsilon_{m+1}^{m+1}} t^{s-3}\, \widetilde{\mathscr{V}}_{m+1}(t)\, dt \cdot$$

Let us now note that

$$\widetilde{\mathscr{V}}_m\left(\varepsilon_m^m\right) \quad = \quad \frac{\varepsilon_m^m}{2}\left(\mathscr{W}(0)+\mathscr{W}\left(2\,\varepsilon_m^m\right)-2\,\mathscr{W}\left(\varepsilon_m^m\right)\right),$$

and

$$\widetilde{\mathscr{V}}_{m+1}\left(\varepsilon_{m+1}^{m+1}\right) \quad = \quad \frac{\varepsilon_{m+1}^{m+1}}{2}\left(\mathscr{W}(0)+\mathscr{W}\left(2\,\varepsilon_{m+1}^{m+1}\right)-2\,\mathscr{W}\left(\varepsilon_{m+1}^{m+1}\right)\right) \cdot$$

Since

$$\varepsilon_{m+1}^{m+1} = \frac{1}{N_b}\,\varepsilon_m^m$$

and thanks to the scaling relation satisfied by $\mathscr{W}$,

$$\mathscr{W}\left(\varepsilon_{m+1}^{m+1}\right) = \mathscr{W}\left(\frac{1}{N_b}\,\varepsilon_m^m\right) = \lambda\,\mathscr{W}\left(\varepsilon_m^m\right) + \cos\left(2\,\pi\,\varepsilon_m^m\right)\,,$$

and

$$\mathscr{W}\left(2\,\varepsilon_{m+1}^{m+1}\right) = \mathscr{W}\left(\frac{2}{N_b}\,\varepsilon_m^m\right) = \lambda\,\mathscr{W}\left(2\,\varepsilon_m^m\right) + \cos\left(4\,\pi\,\varepsilon_m^m\right)\,,$$

we can deduce that

$$\widetilde{\mathscr{V}}_{m+1}(\varepsilon_{m+1}^{m+1}) = \frac{\lambda}{N_b}\,\mathscr{V}_m(N_b\,\varepsilon_{m+1}^{m+1}) + \frac{1}{N_b}\,\frac{N_b\,\varepsilon_{m+1}^{m+1}}{2}\,(1-\lambda) + \frac{N_b\,\varepsilon_{m+1}^{m+1}}{2}\,\left(\cos\left(4\,\pi\,N_b\,\varepsilon_{m+1}^{m+1}\right) - 2\,\cos\left(2\,\pi\,N_b\,\varepsilon_{m+1}^{m+1}\right)\right)\,.$$

ii. We now assume that $(\mathscr{E}_m)$ holds for all $m \geq m_0$.

a. We denote by $\mathscr{P}\left(\tilde{\zeta}^e_m\right) \subset \mathbb{C}$ the set of poles of the zeta function $\tilde{\zeta}^e_m$, and by $\mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right) \subset \mathbb{C}$ the set of poles of the zeta function $\tilde{\zeta}^e_{m_0}$.

We can note that

$$\mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right) \subset \{s \in \mathbb{C}\,, \mathscr{R}e(s) < 2\} \subset \{s \in \mathbb{C}\,, \mathscr{R}e(s) < 3\} \cdot$$

We set

$$\mathscr{U}^+ = \left(\mathbb{C} \setminus \mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right)\right) \cap \{s \in \mathbb{C}\,, \mathscr{R}e(s) < 1\} \cdot$$

$$\left(\text{resp., } \mathscr{U}^- = \left(\mathbb{C} \setminus \mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right)\right) \cap \{s \in \mathbb{C}\,, 1 < \mathscr{R}e(s) < 3\}\right)$$

Then, the series

$$\sum_{m\geq m_0}\left(N_b^{3-s}\,\tilde{\zeta}_m(s)+\frac{1}{2}\,(1-\lambda)\,N_b^{3-s}\,\frac{\varepsilon_m^{m\,(s-1)}}{s-1}+N_b^{3-s}\int_0^{\varepsilon_m^m} t^{s-2}\,\frac{1}{2\,N_b}\,\mathscr{R}e\left(e^{i\,4\,\pi\,t}-2\,e^{i\,2\,\pi\,t}\right)dt\right)$$

is (locally) normally convergent, and, hence, uniformly convergent on

$$\mathscr{U}^{+}=\left(\mathbb{C}\setminus\mathscr{P}\left(\tilde{\zeta}_{m_0}\right)\right)\cap\{s\in\mathbb{C}\,,\,\mathscr{R}e(s)<1\}$$

$$\left(\text{resp., on } \mathscr{U}^{-}=\left(\mathbb{C}\setminus\mathscr{P}\left(\tilde{\zeta}_{m_0}\right)\right)\cap\{s\in\mathbb{C}\,,\,1<\mathscr{R}e(s)<3\}\right)$$

This ensures **the existence of the limit fractal zeta function**, i.e., the fractal zeta function associated with the Weierstrass Curve $\Gamma_{\mathscr{W}}$, given by

$$\tilde{\zeta}^{e}_{\mathscr{W}}(s)=\lim_{m\to\infty}\tilde{\zeta}^{e}_m(s)=\sum_{m\geq m_0}N_b^{3-s}\,\tilde{\zeta}_m(s)+\frac{1}{2}\,(1-\lambda)\,N_b^{3-s}\,\frac{\varepsilon_m^{m\,(s-1)}}{s-1}+N_b^{3-s}\int_0^{\varepsilon_m^m} t^{s-2}\,\frac{1}{2\,N_b}\,\mathscr{R}e\left(e^{i\,4\,\pi\,t}-2\,e^{i\,2\,\pi\,t}\right)dt\,\cdot$$

More precisely, if $\mathbb{P}^1(\mathbb{C})$ denotes the Riemann sphere, we can show that, for the chordal metric, defined, for all $(z_1, z_2) \in \left(\mathbb{P}^1(\mathbb{C})\right)^2$ by

$$\|z_1, z_2\| = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1^2|}\,\sqrt{1 + |z_2^2|}} \,,$$

we have, thanks to the uniform convergence of the series,

$$\lim_{m \to \infty} \left\|\tilde{\zeta}_m^e, \tilde{\zeta}_{\mathscr{W}}^e\right\| = 0 \cdot$$

Indeed, for any $\eta > 0$, we can choose $m_0 \in \mathbb{N}^\star$ such that, for all $s \in \mathbb{P}^1(\mathbb{C})$, we have that

$$\left|\tilde{\zeta}_m^e(s) - \tilde{\zeta}_{\mathscr{W}}^e(s)\right| \leq \eta \,,$$

and, hence, for all $s \in \mathbb{P}^1(\mathbb{C})$,

$$\left\|\tilde{\zeta}_m^e(s), \tilde{\zeta}_{\mathscr{W}}(s)\right\| \leq \left|\tilde{\zeta}_m^e(s) - \tilde{\zeta}_{\mathscr{W}}^e(s)\right| \leq \eta \cdot$$

The sum of this series, i.e., the (uniform) limit fractal zeta function $\tilde{\zeta}^e_{\mathscr{W}}$, is holomorphic on $\mathscr{U}^+$ (resp., on $\mathscr{U}^-$). We can then deduce that, for all $m \geq m_0$, the zeta function $\tilde{\zeta}^e_m$ is meromorphic on $\mathbb{C} \setminus \{s \in \mathbb{C}\,, \mathscr{R}e(s) = 1\}$, and that **its poles in $\mathbb{C} \setminus \{s \in \mathbb{C}\,, \mathscr{R}e(s) = 1\}$** are exactly **the same as the poles of $\tilde{\zeta}^e_{m_0}$**. Moreover, the results obtained in[XXIII] for the sequence of tube zeta functions associated with the Weierstrass IFD, which admit a meromorphic continuation to all of $\mathbb{C}$, obviously hold for the sequence of polyhedral tube zeta functions: hence, $\tilde{\zeta}^e_m$ is meromorphic on $\mathbb{C}$, and its poles belong to $\mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right)$. Consequently, the poles of $\tilde{\zeta}^e_m$ are simple, and are the same as the poles of $\tilde{\zeta}^e_{m_0}$:

$$\mathscr{P}\left(\tilde{\zeta}^e_m\right) = \mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right) \cdot$$

[XXIII] Claire David and Michel L. Lapidus. *Fractal Complex Dimensions and Cohomology of the Weierstrass Curve*. 2022.

b. Let us now denote by $\mathscr{P}\left(\tilde{\zeta}^e_{\mathscr{W}}\right) \subset \mathbb{C}$ the set of poles of the limit fractal zeta function $\tilde{\zeta}^e_{\mathscr{W}}$. By applying Theorem 3.14 given page 82 in[XXIV], we then deduce that

$$\lim_{m\to\infty} \mathscr{P}\left(\tilde{\zeta}^e_m\right) = \mathscr{P}\left(\tilde{\zeta}^e_{\mathscr{W}}\right) \cdot$$

Since, for all $m \geq m_0$,

$$\mathscr{P}\left(\tilde{\zeta}^e_m\right) = \mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right) \,,$$

this ensures that

$$\mathscr{P}\left(\tilde{\zeta}^e_{\mathscr{W}}\right) = \mathscr{P}\left(\tilde{\zeta}^e_{m_0}\right) \cdot$$

Hence, as desired, the poles of the limit of the fractal zeta function $\tilde{\zeta}^e_{\mathscr{W}}$ are simple, and are the same as the poles of $\tilde{\zeta}^e_{m_0}$.

[XXIV] Michel L. Lapidus and Machiel van Frankenhuijsen. *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*. Springer Monographs in Mathematics. Springer, New York, second revised and enlarged edition (of the 2006 edition), 2013, pp. xxvi+567.

From the m^{th}-Prefractal Polyhedral Zeta Function, to the m^{th} Tube Zeta Function

Given $m \in \mathbb{N}$, the Lebesgue measure of the tubularneighborhood $\mathscr{D}\left(\Gamma_{\mathscr{W}_m}, \varepsilon_m^m\right)$ can be connected to the Lebesgue measure of the (m, ε_m^m) polygonal neighborhood $\mathscr{V}_m\left(\varepsilon_m^m\right)$ by means of the following relation,

$$\mathscr{V}_m\left(\varepsilon_m^m\right) = \mathscr{V}_m^{tube}\left(\varepsilon_m^m\right) + \mathscr{R}_m \quad \text{where} \quad \mathscr{V}_m^{tube}\left(\varepsilon_m^m\right) = \mu_{\mathscr{L}}\left(\mathscr{D}\left(\Gamma_{\mathscr{W}_m}\right)\right),$$

and where the sequence of remainders $(\mathscr{R}_m)_{m \geq m_0}$ (locally) uniformly converges to 0.

This ensures, for the associated fractal zeta function

$$s \mapsto \int_0^{\varepsilon_m^m} t^{s-3}\, \mathscr{R}_m(t)\, dt\,,$$

that

$$\lim_{m \to \infty} \int_0^{\varepsilon_m^m} t^{s-3}\, \mathscr{R}_m(t)\, dt = 0 \cdot$$

Theorem: Fractal Tube Formula for The Weierstrass IFD

Given $m \in \mathbb{N}$ sufficiently large, the *tubular volume* $\mathcal{V}_{\mathcal{W}}(\varepsilon_m^m)$, or two-dimensional Lebesgue measure of the ε_m^m-neighborhood of the m^{th} prefractal graph $\Gamma_{\mathcal{W}_m}$,

$$\mathscr{D}\left(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m\right) = \left\{M = (x,y) \in \mathbb{R}^2, d\left(M, \Gamma_{\mathcal{W}_{m(\varepsilon_m^m)}}\right) \leq \varepsilon_m^m\right\},$$

is given by

$$\begin{aligned}\mathcal{V}_{\mathcal{W}}(\varepsilon_m^m) \quad = \quad & \sum_{\ell \in \mathbb{Z},\, k \in \mathbb{N}} f_{\ell,k,\text{Rectangles}} \left(\varepsilon_m^m\right)^{2-D_{\mathcal{W}}+k\,(2-D_{\mathcal{W}})-i\,\ell\,p} \\ & + \sum_{\ell \in \mathbb{Z},\, k \in \mathbb{N}} \left(f_{\ell,k,\text{wedges},1} \left(\varepsilon_m^m\right)^{3-i\,\ell\,p} + f_{\ell,k,\text{wedges},2} \left(\varepsilon_m^m\right)^{1+2\,k-i\,\ell\,p} + f_{\ell,k,\text{wedges},3} \left(\varepsilon_m^m\right) 5 \right. \\ & + \sum_{\ell \in \mathbb{Z},\, k \in \mathbb{N}} f_{\ell,k,\text{triangles, parallelograms}} \left(\varepsilon_m^m\right)^{2-i\,\ell\,p} + \pi \left(\varepsilon_m^m\right)^2 - \frac{\pi \left(\varepsilon_m^m\right)^4}{2},\end{aligned}$$

where the notation $f_{\ell,k,\text{Rectangles}}$, $f_{\ell,k,\text{wedges},\ell}$, $1 \leq \ell \leq 3$, and $f_{\ell,k,\text{triangles, parallelograms}}$, respectively account for the coefficients associated to the sums corresponding to the contribution of the rectangles, wedges, triangles and parallelograms.

Theorem: Local and Global Effective Tube Zeta Function for the Weierstrass IFD

The *global tube zeta function* associated to the Weierstrass IFDs, $\tilde{\zeta}_{\mathscr{W}}$, defined by analogy with the work in[XXV], admits a meromorphic continuation to all of $\mathbb{C}$, and is given, for any complex number s, by:

$$\tilde{\zeta}^{e}_{\mathscr{W}}(s) = \lim_{m \to \infty} \tilde{\zeta}^{e}_{m,\mathscr{W}}(s),$$

[XXV] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

where, for all $m \in \mathbb{N}$ sufficiently large, the *local tube zeta function* $\tilde{\zeta}_{m,\mathscr{W}}$ is given, for any complex number s, by

$$\tilde{\zeta}^e_{m,\mathscr{W}}(s) = \sum_{\ell\in\mathbb{Z},\, k\in\mathbb{N}} f_{\ell,k,\text{Rectangles}} \frac{\left(\varepsilon_m^m\right)^{s-D_{\mathscr{W}}+k\,(2-D_{\mathscr{W}})-i\,\ell\,p}}{s-D_{\mathscr{W}}+k\,(2-D_{\mathscr{W}})-i\,\ell\,p}$$

$$+\sum_{\ell\in\mathbb{Z},\, k\in\mathbb{N}} \left\{ f_{\ell,k,\text{wedges},1} \frac{\left(\varepsilon_m^m\right)^{s+1-i\,\ell\,p}}{s+1-i\,\ell\,p} + f_{\ell,k,\text{wedges},2} \frac{\left(\varepsilon_m^m\right)^{s+2\,k-1-i\,\ell\,p}}{s+2\,k-1-i\,\ell\,p} + f_{\ell,k,\text{w}}\right.$$

$$+\sum_{\ell\in\mathbb{Z},\, k\in\mathbb{N}} f_{\ell,k,\text{triangles, parallelograms}} \frac{\left(\varepsilon_m^m\right)^{s-1-i\,\ell\,p}}{s-1-i\,\ell\,p} + \frac{\pi\,\left(\varepsilon_m^m\right)^{s}}{s} - \frac{\pi\,\left(\varepsilon_m^m\right)^{s+2}}{4\,(s+2)}\,.$$

Corollary: Local and Global Distance Zeta Function for the Weierstrass Iterated Fractal Drums

According to the functional equation given in[XXVI] (Th. 2.2.1., page 112), the *global effective distance zeta function* $\zeta^e_{\mathscr{W}}$ is given, for any complex number s, by:

$$\zeta^e_{\mathscr{W}}(s) = \lim_{m\to\infty} \zeta^e_{m,\mathscr{W}}(s)\,,$$

where, for all $m \in \mathbb{N}$ sufficiently large, the *local distance zeta function* $\zeta^e_{m,\mathscr{W}}$ is given, for any complex number s, by

$$\zeta^e_{m,\mathscr{W}}(s) \quad = \quad \left(\varepsilon^m_m\right)^{s-2} \mathscr{V}_m(\varepsilon^m_m) + (2-s)\,\tilde{\zeta}_{\mathscr{W}_m}(s) \qquad \forall\, s \in \mathbb{C}\,.$$

For all $m \in \mathbb{N}$ sufficiently large, the distance zeta function $\zeta^e_{m,\mathscr{W}}$ admits a meromorphic continuation to all of $\mathbb{C}$, given by the last equality just above.

[XXVI] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

Remark

The Complex Dimensions – i.e., the poles of $\zeta^{e}_{m,\mathscr{W}}$, or, equivalently, of $\tilde{\zeta}^{e}_{m,\mathscr{W}}$

are independent of the choice of the parameter $\varepsilon^{\boldsymbol{m}}_{\boldsymbol{m}}$

(see the general theory developed in [XXVII])

This comes from the fact that, for $0 < \varepsilon^{\boldsymbol{m}}_{\boldsymbol{m},1} < \varepsilon^{\boldsymbol{m}}_{\boldsymbol{m},2}$,

$\zeta^{\boldsymbol{e}}_{\boldsymbol{m},\mathscr{W},\varepsilon^{m}_{m,1}} - \zeta^{\boldsymbol{e}}_{\boldsymbol{m},\mathscr{W},\varepsilon^{m}_{m,2}}$ **is an entire function.**

[XXVII] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

Theorem: Complex Dimensions of the $\mathscr{W}$ IFD[XXIX]

The **possible** **Complex Dimensions** of the Weierstrass IFD are **all simple**, and given as follows:

$$D_{\mathscr{W}} - k\,(2 - D_{\mathscr{W}}) + i\,\ell\,p \quad , \quad k \in \mathbb{N}\,, \ell \in \mathbb{Z}\,,$$

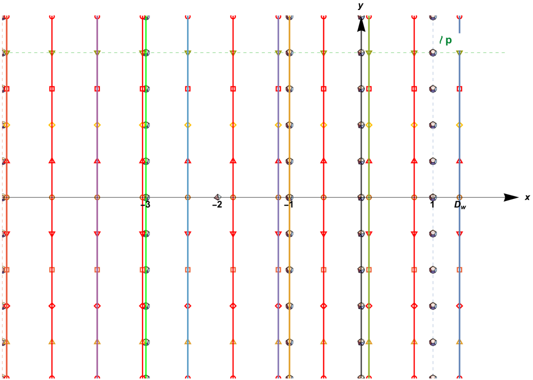
$$1 - 2\,k + i\,\ell\,p \quad , \quad k \in \mathbb{N}\,, \ell \in \mathbb{Z}, \text{ along with } -2 \text{ and } 0\cdot$$

The **one-periodic functions** (with respect to $\ln_{N_b}\left(\varepsilon_m^m\right)^{-1}$), resp. associated to the values $D_{\mathscr{W}} - k\,(2 - D_{\mathscr{W}})$, $k \in \mathbb{N}$, are **nonconstant**. In addition, all of their Fourier coefficients are nonzero, which implies that **there are infinitely many Complex Dimensions that are nonreal**, including those with maximal real part $D_{\mathscr{W}}$, which are **the principal Complex Dimensions** (see [XXVIII]). They give rise to **geometric oscillations with the largest amplitude**, **in the fractal tube formula**.

[XXVIII] Michel L. Lapidus, Goran Radunović, and Darko Žubrinić. *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*. Springer Monographs in Mathematics. Springer, New York, 2017, pp. xl+655.

[XXIX] Claire David and Michel L. Lapidus. *Weierstrass fractal drums - I - A glimpse of complex dimensions*. 2022.

Complex Dimensions of the Weierstrass IFD



The nonzero Complex Dimensions are periodically distributed (with the same period $p = \frac{2\pi}{\ln N_b}$, the oscillatory period of $\Gamma_{\mathscr{W}}$) along countably many vertical lines, with abscissae $D_{\mathscr{W}} - k\,(2 - D_{\mathscr{W}})$ and $1 - 2k$, where $k \in \mathbb{N}$ is arbitrary. In addition, 0 and -2 are Complex Dimensions of $\Gamma_{\mathscr{W}}$.

Theorem: Complex Dimensions of the Weierstrass Curve

The Complex Dimensions of The Weierstrass Curve are all simple, and given as follows:

$$D_{\mathscr{W}} - k\,(2 - D_{\mathscr{W}}) + i\,\ell\,p \quad , \quad \text{with } k \in \mathbb{N}\,,\, \ell \in \mathbb{Z}\,,$$

$$1 - 2\,k + i\,\ell\,p \quad , \quad \text{with } k \in \mathbb{N}\,,\, \ell \in \mathbb{Z}, \text{ along with } 0 \text{ and } 1 \cdot$$

Proof

i. First, there exists an integer $m_0 \in \mathbb{N}$ such that the poles of the limit effective fractal zeta function $\tilde{\zeta}^e_{\mathcal{W}}$, i.e., the fractal zeta function associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$, are the same as the poles of the fractal zeta function $\tilde{\zeta}^e_{m_0}$.

ii. Second, we have showed the poles of the limit fractal zeta function $\tilde{\zeta}^e_{\mathcal{W}}$ are also the same as the poles of the tube fractal zeta function $\tilde{\zeta}^{e,tube}_{m_0}$.

iii. We then dispose of the results obtained in[XXX], which give the values of the poles of the tube fractal zeta function $\tilde{\zeta}^{e,tube}_{m_0}$.

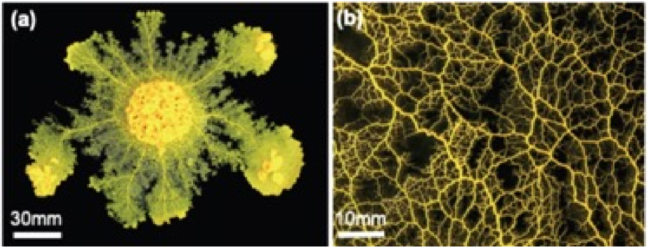
[XXX] Claire David and Michel L. Lapidus. *Weierstrass fractal drums - I - A glimpse of complex dimensions*. 2022.

Connections with Real Life

Connections with Real Life

⇝ Nature produces many **fractal-like structures**. Until now, the **tools of fractal geometry** have been little used to model the **morphogenesis** of these living forms.

⇝ The acellular model organism **Physarum polycephalum** grows in a **network and fractal branched way**.



(a) P. polycephalum plasmodium. (b) Vein network.
© A. Dussutour & C. Oettmeier.

⇝ The change of shape in **Physarum polycephalum** corresponds to a **change of fractal (complex) dimensions** (undergoing work with A. Dussutour, H. Henni, C. Godin).

⇝ Just as in our **mathematical theory**.

⇝ What is **the growth law**?

⇝ Can we find **the underlying variational principle**?

Forthcoming: The Magnitude

⇝ Counterpart of the **(topological) Euler characteristic**[XXXI].

⇝ New method for **numerically determining the Complex Dimensions of a fractal**[XXXII].

⇝ Also **connected** to **the polyhedral measure**.

[XXXI] Tom Leinster. "The magnitude of metric spaces". In: *Documenta Mathematica* 18 (2013), pp. 857–905. ISSN: 1431-0635.

[XXXII] Claire David and Michel L. Lapidus. *Fractal Complex Dimensions ~ A Bridge to Magnitude*. 2023.