Fractal zeta functions - a short overview

Goran Radunović

Bifurcations and Fractal Zeta Functions of Orbits

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Joint work with:
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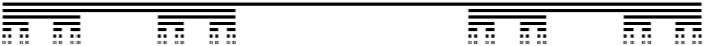

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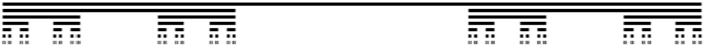

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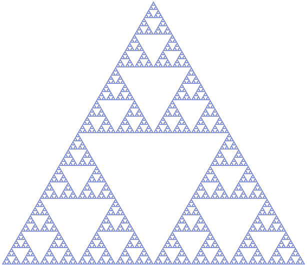

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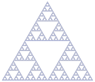

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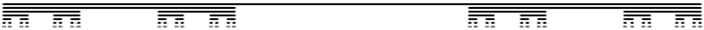

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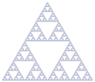

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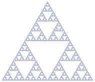

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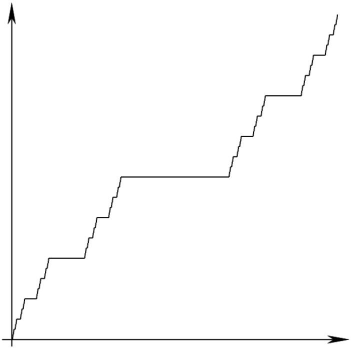
- Mandelbrot: A set is fractal if its fractal dimension exceeds its topological dimension.
- None of the above dimensions give a completely satisfactory definition of a fractal.

Some more examples



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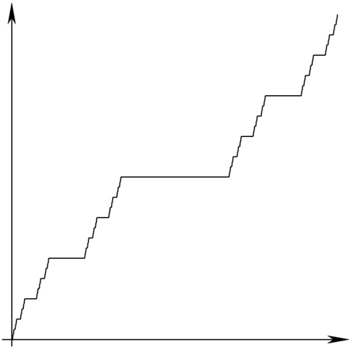

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All of the known fractal dimensions are equal to 1, i.e., to its topological dimension.

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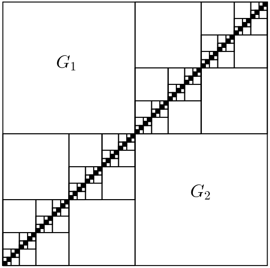

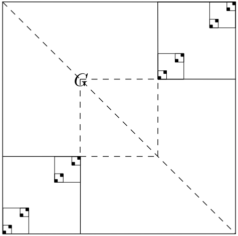

Figure: Left: The $1/2$-square fractal. Right: The $1/3$-square fractal.

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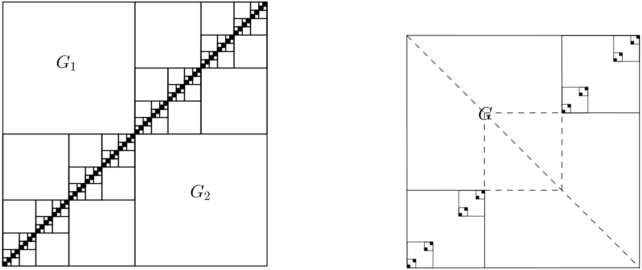

Figure: **Left: The $1/2$-square fractal. Right: The $1/3$-square fractal.**

The Hausdorff and Minkowski dimensions equal to 1 which is also their topological dimension.

The Minkowski content and dimension

- $\emptyset \neq A \subset \mathbb{R}^N$
- δ-neighbourhood of A:

$$A_\delta = \{x \in \mathbb{R}^N : d(x, A) < \delta\}$$

- r-**dimensional Minkowski content of** A:

$$\mathcal{M}^r(A) := \lim_{\delta \to 0^+} \frac{|A_\delta|}{\delta^{N-r}}$$

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- **Minkowski dimension of** A:

$$\dim_B A = \inf\{r \in \mathbb{R} \ : \ \mathcal{M}^r(A) = 0\}$$
$$= \sup\{r \in \mathbb{R} \ : \ \mathcal{M}^r(A) = \infty\}$$

The geometric zeta function [Lapidus, van Frankenhuijsen, Pomerance, Maier]

- **fractal string**: $\mathcal{L} = (\ell_j)_{j \geq 1}$ $\ell_j \searrow 0$
- $A_{\mathcal{L}} := \{a_k := \sum_{j \geq k} \ell_j : k \geq 1\}$
- **geometric zeta function**: $$\zeta_{\mathcal{L}}(s) := \sum_{j=1}^{\infty} {\ell_j}^s$$

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The set of complex dimensions: $\left\{\log_3 2 + \frac{2\pi \mathbb{i}\mathbb{Z}}{\log 3}\right\}$.

The Distance Zeta Function - generalization to higher dimensions [LaRaZu]

- the **distance zeta function of** $A \subset \mathbb{R}^N$:

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- $\zeta_{A_\mathcal{L}}(s) = \frac{2^{1-s}}{s}\zeta_\mathcal{L}(s) + \frac{2\delta^s}{s}$, given a large enough $\delta > 0$

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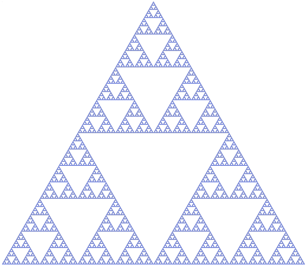
Assume ζ_A can be meromorphically extended to $W \subseteq \mathbb{C}$.
The **set of complex dimensions** of A **visible in** W:

$$\mathcal{P}(\zeta_A, W) := \Big\{\omega \in W : \omega \text{ is a pole of } \zeta_A\Big\}.$$

Complex dimensions of the Sierpiński gasket



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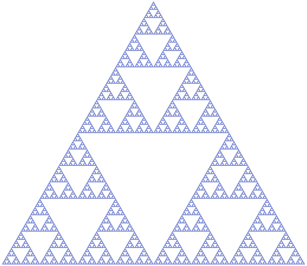

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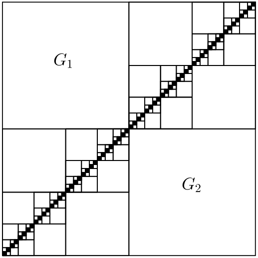
$$\mathcal{P}(\zeta_A) = \{0, 1\} \cup \left(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right)$$

Complex dimensions of the $1/2$-square fractal



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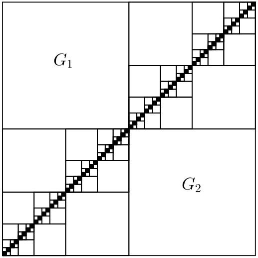

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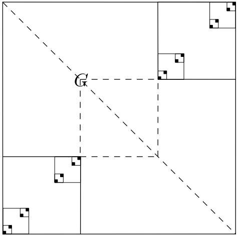
Complex dimensions of the $1/3$-square fractal



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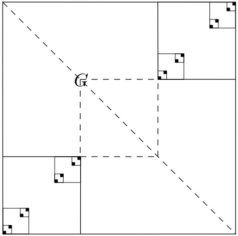

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Relative fractal drum (A, Ω)

- $\emptyset \neq A \subset \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$, Lebesgue measurable, i.e., $|\Omega| < \infty$
- **upper** r**-dimensional Minkowski content of** (A, Ω):

$$\overline{\mathcal{M}}^r(A, \Omega) := \limsup_{\delta \to 0^+} \frac{|A_\delta \cap \Omega|}{\delta^{N-r}}$$

- **upper Minkowski dimension of** (A, Ω):

$$\overline{\dim}_B(A, \Omega) = \inf\{r \in \mathbb{R} \ : \ \overline{\mathcal{M}}^r(A, \Omega) = 0\}$$

- **lower Minkowski content** and **dimension** defined via lim inf

Minkowski measurability

- $\underline{\dim}_B(A,\Omega) = \overline{\dim}_B(A,\Omega)$ $\Rightarrow$ $\exists \dim_B(A,\Omega)$
- if $\exists D \in \mathbb{R}$ such that

$$0 < \underline{\mathcal{M}}^D(A,\Omega) = \overline{\mathcal{M}}^D(A,\Omega) < \infty,$$

  we say (A,Ω) is **Minkowski measurable**; in that case $D = \dim_B(A,\Omega)$
- if the above inequalities are not satisfied for D, we call (A,Ω) **Minkowski degenerated**

The relative distance zeta function

- (A, Ω) RFD in $\mathbb{R}^N$, $s \in \mathbb{C}$ and **fix** $\delta > 0$
- the **distance zeta function of** (A, Ω):

$$\zeta_{A,\Omega}(s;\delta) := \int_{A_\delta \cap \Omega} d(x, A)^{s-N}\, dx$$

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- dependence on δ is not essential
- the **complex dimensions of** (A, Ω) are defined as the poles of $\zeta_{A,\Omega}$
- take Ω to be an open neighborhood of A in order to recover the classical ζ_A

The relative tube zeta function

(A,Ω) an RFD in $\mathbb{R}^N$ and fix $\delta > 0$

- the **tube zeta function of** (A,Ω):

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- dependence on δ is inessential
- analogous holomorphicity theorem holds for $\widetilde{\zeta}_{A,\Omega}(s;\delta)$

- a functional equation connecting the two zeta functions:

$$\zeta_{A,\Omega}(s;\delta) = \delta^{s-N} |A_\delta \cap \Omega| + (N - s) \widetilde{\zeta}_{A,\Omega}(s;\delta)$$

Fractal tube formulas for relative fractal drums

- An asymptotic (Steiner-type) formula for the **tube function** $t \mapsto |A_t \cap \Omega|$ as $t \to 0^+$ in terms of $\zeta_{A,\Omega}$.

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Theorem (Simplified pointwise formula with error term)

• $\alpha < \overline{\dim}_B(A,\Omega) < N$; $\zeta_{A,\Omega}$ satisfies suitable rational decay (d-**languidity**) on the half-plane $\mathbf{W} := \{\mathrm{Re}\, s > \alpha\}$, then:

$$|A_t \cap \Omega| = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega}, \mathbf{W})} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_{A,\Omega}(s), \omega\right) + O(t^{N-\alpha}).$$

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- if we allow polynomial growth of $\zeta_{A,\Omega}$, in general, we get a tube formula in the sense of Schwartz distributions

Fractal tube formulas for relative fractal drums

- An asymptotic formula for the **tube function** $t \mapsto V_{A,\Omega}(t) := |A_t \cap \Omega|$ as $t \to 0^+$ in terms of $\zeta_{A,\Omega}$.

Theorem (Case of simple poles)

- In case the the fractal zeta function has only simple poles:

$$V_{A,\Omega}(t) = \sum_{\omega \in \mathcal{P}(\zeta_{A,\Omega},\mathbf{W})} \frac{t^{N-\omega}}{N-\omega} \operatorname{res}(\zeta_{A,\Omega}(s), \omega) + O(t^{N-\alpha}).$$

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- a pole ω of order m generates terms of type $t^{N-\omega}(-\log t)^{k-1}$ for $k = 1, \ldots, m$
- if $\omega \in \mathbb{C} \setminus \mathbb{R}$ then the term $t^{N-\omega} = t^{N-\operatorname{Re}\omega} \mathrm{e}^{-\mathbb{i} \operatorname{Im}\omega \log t}$ introduces oscillations in the order $t^{N-\operatorname{Re}\omega}$ which are multiplicative periodic with period $T = \mathrm{e}^{2\pi/\operatorname{Im}\omega}$

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In that case:

$$\mathcal{M}^D(A, \Omega) = \frac{\mathrm{res}(\zeta_{A,\Omega}, D)}{N - D}$$

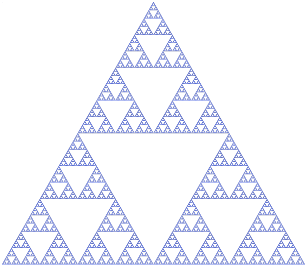
The Minkowski measurability criterion

- $(a) \Rightarrow (b)$: from the distributional tube formula and the **Uniqueness theorem for almost periodic distributions** due to **Schwartz**

- $(b) \Rightarrow (a)$: a consequence of a **Tauberian theorem** due to **Wiener** and **Pitt** (conditions can be considerably weakened)

- the assumption $D < N$ can be removed by appropriately embedding the RFD in $\mathbb{R}^{N+1}$

Figure: The Sierpiński gasket



- an example of a **self-similar fractal spray** with a generator G being an open equilateral triangle and with **scaling ratios** $r_1 = r_2 = r_3 = 1/2$
- $(A, \Omega) = (\partial G, G) \sqcup \bigsqcup_{j=1}^{3} (r_j A, r_j \Omega)$

Fractal tube formula for The Sierpiński gasket

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1}$$

By letting $\omega_k := \log_2 3 + \mathbf{p}k\mathbb{i}$ and $\mathbf{p} := 2\pi/\log 2$ we have that

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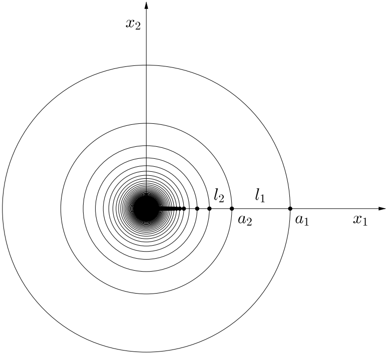
$$\begin{aligned}
|A_t| &= \sum_{\omega\in\mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta),\omega\right) \\
&= t^{2-\log_2 3}\frac{6\sqrt{3}}{\log 2}\sum_{k=-\infty}^{+\infty}\frac{(4\sqrt{3})^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k-1)\omega_k} + \left(\frac{3\sqrt{3}}{2}+\pi\right)t^2, \\
&= t^{2-\log_2 3}H(\log_2 t) + \left(\frac{3\sqrt{3}}{2}+\pi\right)t^2
\end{aligned}$$

valid pointwise for all $t \in (0, 1/2\sqrt{3})$; $H\colon \mathbb{R} \to \mathbb{R}$ is 1-periodic, $0 < \min H = \underline{\mathcal{M}}^{2-\log_2 3}(A) < \overline{\mathcal{M}}^{2-\log_2 3}(A) = \max H < +\infty$

The fractal nest generated by the a-string



$a > 0$, $a_j := j^{-a}$, $\ell_j := j^{-a} - (j+1)^{-a}$, $\Omega := B_{a_1}(0)$

$$\zeta_{A_a,\Omega}(s) = \frac{2^{2-s}\pi}{s-1} \sum_{j=1}^{\infty} \ell_j^{s-1}(a_j + a_{j+1})$$

Fractal tube formula for the fractal nest generated by the a-string

Example

$$\mathcal{P}(\zeta_{A_a,\Omega}) \subseteq \left\{1, \frac{2}{a+1}, \frac{1}{a+1}\right\} \cup \left\{-\frac{m}{a+1} : m \in \mathbb{N}\right\}$$

$a \neq 1,\ D := \frac{2}{1+a} \Rightarrow$

$$|(A_a)_t \cap \Omega| = \frac{2^{2-D} D\pi}{(2-D)(D-1)} a^{D-1} t^{2-D} + 2\pi\big(2\zeta(a) - 1\big)t + O\big(t^{2-\frac{1}{a+1}}\big), \text{ as } t \to 0^+$$

Fractal tube formula for the fractal nest generated by the a-string

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$$= 2\pi t(-\log t) + \mathrm{const} \cdot t + o(t) \quad \text{as } t \to 0^+$$

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- a pole ω of order m generates terms of type $t^{N-\omega}(-\log t)^{k-1}$ for $k = 1, \ldots, m$ in the fractal tube formula

Some more examples

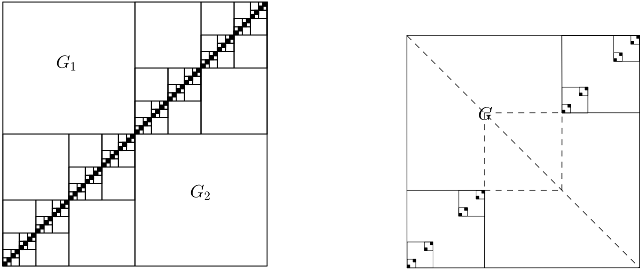

Figure: **Left: The $1/2$-square fractal. Right: The $1/3$-square fractal.**

The Hausdorff and Minkowski dimensions equal to 1 which is also their topological dimension.

Fractal tube formula for the $1/2$-square fractal

$$\zeta_A(s) = \frac{2^{-s}}{s(s-1)(2^s-2)} + \frac{4}{s-1} + \frac{2\pi}{s}, \tag{5}$$

$$D(\zeta_A) = 1, \quad \mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) = \{0\} \cup \left(1 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\right). \tag{6}$$

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$$\begin{aligned} |A_t| &= \sum_{\omega \in \mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s), \omega\right) \\ &= \frac{1}{4\log 2} t \log t^{-1} + t\, G\left(\log_2 (4t)^{-1}\right) + \frac{1+2\pi}{2} t^2, \end{aligned} \tag{7}$$

valid for all $t \in (0, 1/2)$, where G is a nonconstant 1-periodic function on $\mathbb{R}$ bounded away from zero and ∞.
The $1/2$-square fractal is **critically fractal** in dimension 1.

Fractal tube formula for the $1/3$-square fractal

$$\zeta_A(s) = \frac{2}{s(3^s-2)}\left(\frac{6}{s-1} + Z(s)\right) + \frac{4}{s-1} + \frac{2\pi}{s}, \tag{8}$$

$$\mathcal{P}(\zeta_A) := \mathcal{P}(\zeta_A, \mathbb{C}) \subseteq \{0\} \cup \left(\log_3 2 + \frac{2\pi}{\log 3}\mathbb{i}\mathbb{Z}\right) \cup \{1\}, \tag{9}$$

$$\begin{aligned} |A_t| &= \sum_{\omega\in\mathcal{P}(\zeta_A)} \operatorname{res}\left(\frac{t^{2-s}}{2-s}\zeta_A, \omega\right) \\ &= 16t + t^{2-\log_3 2} G\left(\log_3(3t)^{-1}\right) + \frac{12+\pi}{2}t^2. \end{aligned} \tag{10}$$

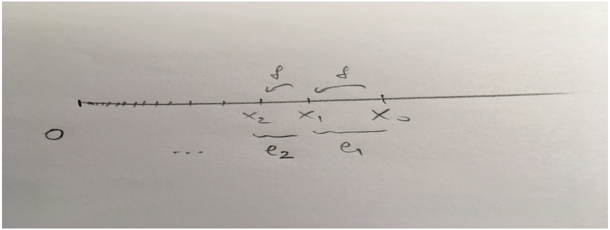
valid for all $t \in (0, 1/\sqrt{2})$, where G is a nonconstant 1-periodic function on $\mathbb{R}$ bounded away from zero and infinity.
The $1/3$-square fractal is **subcritically fractal** in dimension $\omega = \log_3 2 < \dim_B A = 1$.

Parabolic analytic germs (joint with Mardešić and Resman)

- let f be an attracting germ of a diffeo. on $\mathbb{R}$ at a fixed point 0 and let

$$\mathcal{O}_f(x_0) := \{f^{\circ n}(x_0) : \ n \in \mathbb{N}\},$$

be its orbit by f.



- Can one read the formal (or even analytic) class of f from the "fractality" of its one orbit?
- The tube function of the orbit:

$$V_f := V_{f,x_0} : \varepsilon \mapsto |\mathcal{O}_f(x_0)_\varepsilon \cap [0, x_0]|$$

Parabolic analytic germs

- $$f(x) = x - ax^{k+1} + o(x^{k+1}),\ a > 0,\ x \to 0. \qquad (11)$$
- Formal change of variables in the class of formal power series $x + x^2\mathbb{R}[[x]]$ reduces f to a normal form which is a time-one map of a simple vector field:

$$f_0(x) = \mathrm{Exp}\left(-\frac{x^{k+1}}{1-\rho x^k}\frac{d}{dx}\right).\mathrm{id},\ k \in \mathbb{N},\ \rho \in \mathbb{R}. \qquad (12)$$

  Parabolic germs of the type (12) are called *model diffeomorphisms*.
- the pair $(k, \rho) \in \mathbb{N} \times \mathbb{R}$ is called the *formal invariant* of f.

Fractal zeta function for the general non-model case

Arbitrary parabolic germ

$$f(x) = x - ax^{k+1} + o(x^{k+1}) \in \mathrm{Diff}(\mathbb{R}_+, 0)$$

Theorem (B MRR 2020, Complex dimensions for arbitrary parabolic orbits)

$f \in \mathrm{Diff}(\mathbb{R}_+, 0)$, *of formal class* (k, ρ), $k \in \mathbb{N}$, $\rho \in \mathbb{R}$.

1. *The distance zeta function* $\zeta_f(s)$ *can be meromorphically extended to* $\mathbb{C}$.
2. *In any open right half-plane* $W_M := \{\mathrm{Re}\, s > 1 - \frac{M}{k+1}\}$, *where* $M \in \mathbb{N}$, $M > k + 2$, *given as:*

Theorem

For $s \in W_M := \{\operatorname{Re} s > 1 - \frac{M}{k+1}\}$*:*

$$\zeta_f(s) = (1-s)\sum_{m=1}^{k} \frac{a_m}{s - \left(1 - \frac{m}{k+1}\right)} + (1-s)\left(\frac{b_{k+1}(x_0)}{s} + \frac{a_{\rho,k}}{s^2}\right)$$

$$+ (1-s)\sum_{m=k+2}^{M-1} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} \frac{(-1)^p p! \cdot c_{m,p}(x_0)}{\left(s - \left(1 - \frac{m}{k+1}\right)\right)^{p+1}} + g(s),$$

$g(s)$ *holomorphic in* W_M*.*

$*$ the coefficients are real, depending on coeffs. of f and x_0, as noted!

$*$ **higher-order poles** correspond to *logarithmic terms* in the asymptotic expansion of the tube function due to $\rho \neq 0$

Formal class from complex dimensions

Corollary (MRR Formal class of a parabolic germ from complex dimensions)

Let f be a parabolic germ $f(x) = x - ax^{k+1} + o(x^{k+1})$, $a > 0$ from the formal class (k, ρ). Then ζ_f is meromorphic in $\mathbb{C}$ and the formal class is encoded in two complex dimensions:

1. *the simple pole with largest real part, $\omega_1 = 1 - \frac{1}{k+1}$, and its residue:*

$$\mathrm{Res}(\zeta_f(s), \omega_1) = \frac{a_1}{k+1} = \frac{2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}}}{k},$$

2. *the double pole with largest real part, $\omega_{k+1} = 0$, and the residue:*

$$\mathrm{Res}(s \cdot \zeta_f(s), \omega_{k+1}) = a_{\rho,k} = 2\rho \frac{k-1}{k}.$$

Model hyperbolic orbits

- $f_a(x) = ax$, $0 < a < 1$
- $\mathcal{O}_{f_a}(x_0) = \{x_0 a^n : n \in \mathbb{N}_0\}$
- $\mathcal{L}_{f_a} = \{\ell_j = f_a^{\circ j}(x_0) - f_a^{\circ(j+1)}(x_0) = x_0(1-a)a^j : j \in \mathbb{N}_0\}$
- $$\zeta_{f_a}(s) = \frac{2^{1-s}}{s} \sum_{j=0}^{\infty} \ell_j^s = \frac{2^{1-s} x_0^s (1-a)^s}{s} \cdot \frac{1}{1-a^s}$$
- extends meromorphically to all of $\mathbb{C}$ from $\{\operatorname{Re} s > 0\}$
- double pole at $s = 0$ and simple poles at
$$s_k = \frac{2k\pi \mathbb{i}}{\log a},\ \ k \in \mathbb{Z}$$
- $V_f(\varepsilon) = -\frac{2}{\log a}\varepsilon(-\log\varepsilon) + \varepsilon H\left(\log_a \frac{2\varepsilon}{x_0(1-a)}\right)$,
$H : [0, +\infty) \to \mathbb{R}$ is 1-periodic and bounded

Parabolic orbits vs. hyperbolic orbits and fractality

!! parabolic case: oscillations of the coefficients can be smoothened by integration
!! hyperbolic case: the oscillations are mulitiplicative periodic and cannot be smoothened distributionally

(a) parabolic orbits: $\tau_\varepsilon \sim \varepsilon^{-\frac{1}{k+1}}$, $\frac{d}{d\varepsilon}\tau_\varepsilon \sim \varepsilon^{-1-\frac{1}{k+1}}$, where $1+\frac{1}{k+1}>1$

(b) hyperbolic orbits: $\tau_\varepsilon \sim -\log\varepsilon$, $\frac{d}{d\varepsilon}\tau_\varepsilon \sim -\varepsilon^{-1}$

The consequence:
($*$) in the parabolic case **no oscillatory coefficients in the distributional expansion** (seen in poles of zeta function as **no non-real complex dimensions**)
($*$) in the hyperbolic case **oscillatory coefficients remain** (seen in poles of zeta function as **purely imaginary complex dimensions**, similarly as for Cantor sets (LF 2013, LRZ 2017)

? who is fractal ?

Reach zeta functions - joint with S. Winter

$A \subseteq \mathbb{R}^d$ compact $|A| = 0$, then by [HugLastWeil]:

$$|A_\varepsilon| = \sum_{i=0}^{d-1} \omega_{d-i} \int_0^\varepsilon t^{d-i-1} \int_{N(A)} \mathbb{1}\{t < \delta(A,x,u)\} \mu_i(A; d(x,u)) dt.$$

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$N(A) \subseteq A \times \mathbb{S}^{d-1}$ the generalized normal bundle

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$$\delta(A,x,u) = \quad \text{the reach at} \quad (x,u) \in N(A)$$

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Further research directions

- Extending the notion of complex dimensions to include logarithmic and "mixed" singularities points and connecting them with various gauge functions appearing in fractal tube formulas
- Obtaining corresponding tube formulas and gauge-Minkowski measurability criteria (with M. Lapidus)
- Applying the theory to problems from dynamical systems (with M. Resman, P. Mardesic, M. Klimes, R. Huzak)
- Connecting the theory with *fractal curvatures* and *support measures* (with S. Winter)

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