Regularity properties of parallel volume and surface area

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joint work with Jan Rataj

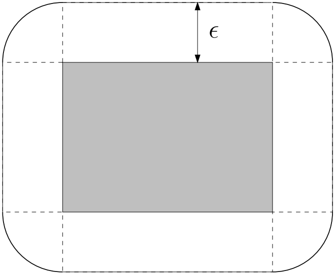
Workshop on Bifurcations and fractal zeta functions of orbits
Zagreb, May 12th, 2023

Parallel sets

- For $A \subset \mathbb{R}^d$ and $r > 0$ let

$$A_r := \left\{ x \in \mathbb{R}^d : d(x, A) < r \right\}$$

be the (open) r-parallel set of A.



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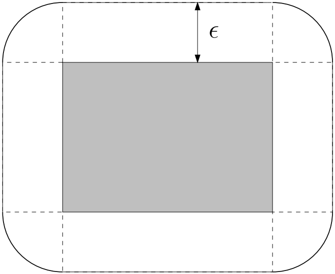

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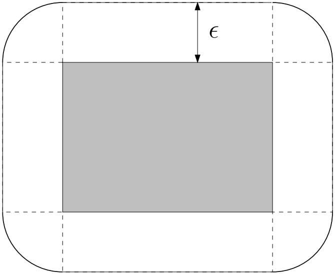

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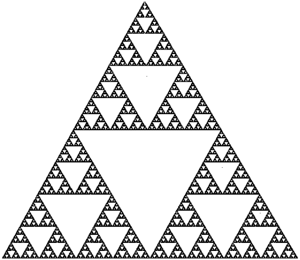

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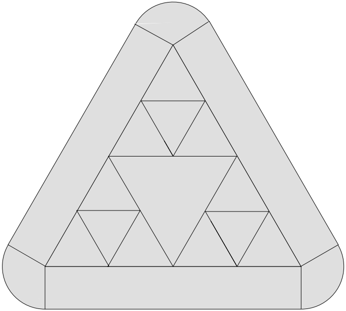

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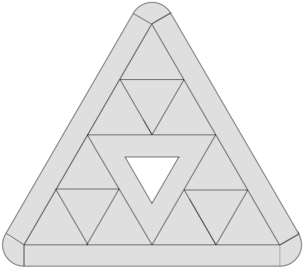

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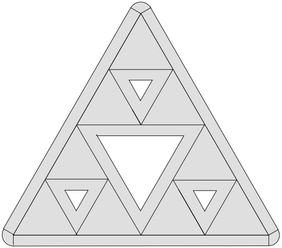

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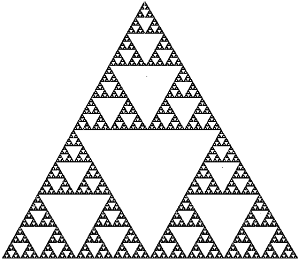

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Parallel volume $V_A : (0, \infty) \to \mathbb{R}$ and parallel surface area $S_A : (0, \infty) \to \mathbb{R}$ of A are defined by

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- $(V_A)'_+(r)$ is right continuous and $(V_A)'_-(r)$ is left continuous

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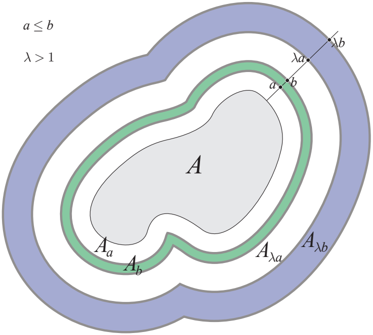

for $f = V_A$: $$V_A(\lambda b) - V_A(\lambda a) \leq \lambda^d \left(V_A(b) - V_A(a)\right).$$

Relations between volume and surface area

- *s*-dim. Minkowski content of $F \subset \mathbb{R}^d$: $\mathcal{M}^s(F) = \lim_{\varepsilon \searrow 0} \varepsilon^{d-s} \lambda_d(F_\varepsilon)$

Stachó's Theorem [Stachó '76]

Let $A \subset \mathbb{R}^d$ be bounded. Then $\mathcal{M}^{d-1}(\partial A_r)$ exists for all $r > 0$ and

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- positive boundary of $F \subset \mathbb{R}^d$: $\partial^+ F = \{x \in \partial F : x \text{ is in the image of } \pi_F\}$

Theorem [Hug, Last, Weil '04]

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Corollary [Rataj, W. '10]

If $(V_A)'(r)$ exists for some $r > 0$, then

$$\mathcal{M}^{d-1}(\partial A_{\leq r}) = \mathcal{M}^{d-1}(\partial A_r) = \mathcal{H}^{d-1}(\partial A_r) = \mathcal{H}^{d-1}(\partial^+ A_r) = (V_A)'(r).$$

The parallel surface measures

A consequence for parallel surface area S_A:

- If $r_0 > 0$ is a differentiability point of V_A, then S_A is continuous at r_0, i.e.,

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Let $A \subset \mathbb{R}^d$ be nonempty and compact and let $r_0 > 0$ be a differentiability point of V_A. Then

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- curvature measures $C_k(A_r, \cdot)$: if $r_0 > 0$ is a regular value of d_A, then

$$C_k(A_r, \cdot) \xrightarrow{w} C_k(A_{r_0}, \cdot) \qquad \text{as } r \to r_0. \qquad \textbf{[W.'15, RWZ'23+]}$$

Critical points and critical values

Let $A \subset \mathbb{R}^d$ be compact.

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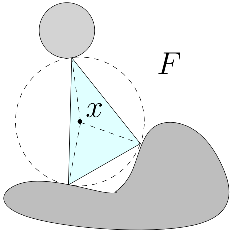

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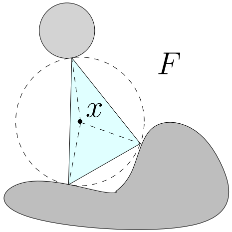

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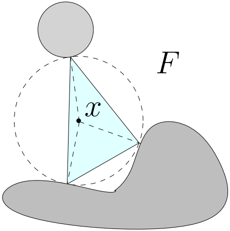

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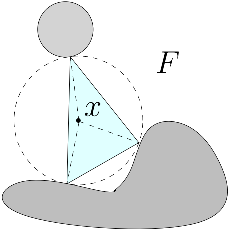
- For regular $r > 0$, the closed complement of A_r, $\overline{(A_r)^c}$ has positive reach, cf. **[Fu 85]**. (In this case, curvature measures of A_r are defined.)



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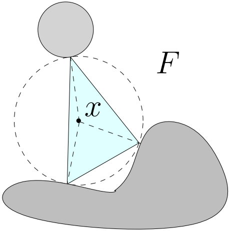

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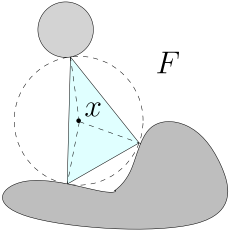

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- $\mathrm{Unp}(A) := \{x \in \mathbb{R}^d : x \text{ has a unique nearest point in } A\}$

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- If $\Pi_A : \mathrm{Unp}(A) \setminus A \to N(A)$, $x \mapsto \left(\pi_A(x), \frac{x - \pi_A(x)}{|x - \pi_A(x)|}\right)$ is the generalized metric projection of A and $\beta \subset \partial A \times \mathbf{S}^{d-1}$, then $\Pi_A^{-1}(\beta)$ is metrically associated with A.

Metrically associated sets

Definition

Let $A \subset \mathbb{R}^d$ be compact. $X \subset \mathbb{R}^d$ is called metrically associated with A, if for any point $x \in X$ there exists a point $a \in A$ so that

- $d(x, a) = d_A(x)$ and
- $]x, a[\subset X$.

Examples:

- A_r is metrically associated with A (for any $r > 0$).
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Lemma [Stachó '76]

Let $A, X \subset \mathbb{R}^d$ and let X be measurable and metrically associated with A. Then the function $t \mapsto f(t) := \lambda_d(A_t \cap X), t > 0$ has the Kneser property.

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$$r \mapsto V_{A,\beta}(r) := \lambda_d(A_r \cap \Pi_A^{-1}(\beta)), r > 0$$

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Proposition [W. '19], [Rataj, W. '23+]

Let $A \subset \mathbb{R}^d$ be nonempty and compact and let $\beta \subset N(A)$ be some Borel set. Then, for any $r > 0$,

$$(V_{A,\beta})'_+(r) = \mathcal{H}^{d-1}(\partial^+ A_r \cap \Pi_A^{-1}(\beta)) \leq \mathcal{H}^{d-1}(\partial A_r \cap \Pi_A^{-1}(\beta)) (= S_A(r, \Pi_A^{-1}(\beta))).$$

Moreover, if V_A is differentiable at r, then '$\leq$' can be replaced by '$=$'.

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Moreover, if V_A is differentiable at r, then '$\leq$' can be replaced by '$=$'.

Note: If V_A is differentiable at some $r > 0$, then $V_{A,\beta}$ is for any $\beta \subset N(A)$.

Idea of Proof for Theorem A

Lemma [Rataj, W. '23+]

Let $A \subset \mathbb{R}^d$ be nonempty and compact and let $r_0 > 0$ be a differentiability point of $r \mapsto V_A(r)$. Then for any Borel set $\beta \subset N(A)$

$$\lim_{r \to r_0} S_A(r, \Pi_A^{-1}(\beta)) = S_A(r_0, \Pi_A^{-1}(\beta)).$$

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- By Prokhorov's Theorem, any sequence of measures in M has a converging subsequence. Let $(r_i)_{i \in \mathbb{N}}$ be a sequence in $(0, R]$ such that $r_i \to r_0$ as $i \to \infty$ and such that for some measure μ

$$\mu_i := S_A(r_i, \cdot) \xrightarrow{w} \mu, \qquad \text{as } i \to \infty.$$

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- One has $\operatorname{spt} \mu \subset \partial A_{r_0}$ (and $\operatorname{spt} \mu \subset \partial A_{r_0}$). Therefore, it is enough to show $\mu(F) = \mu_0(F)$ for any relatively closed set $F \subset \partial A_{r_0} \cap \mathrm{Unp}\,(A)$.

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- One has $\operatorname{spt} \mu \subset \partial A_{r_0}$ (and $\operatorname{spt} \mu \subset \partial A_{r_0}$). Therefore, it is enough to show $\mu(F) = \mu_0(F)$ for any relatively closed set $F \subset \partial A_{r_0} \cap \mathrm{Unp}\,(A)$. (This implies $\mu(\cdot \cap \mathrm{Unp}\,(A)) = \mu_0(\cdot \cap \mathrm{Unp}\,(A))$. Together with $\mu_0(\mathrm{Unp}\,(A)^c) = 0$ and $\mu_i(\mathbb{R}^d) \to \mu_0(\mathbb{R}^d)$ this shows $\mu = \mu_0$.)

Characterization of differentiability of V_A

Theorem B [Hug, Santilli '22, Rataj, W. '23+]

Let $A \subset \mathbb{R}^d$ be compact and $r > 0$. Then the following assertions are equivalent:

(i) the volume function V_A is differentiable at r;

(ii) $\mathcal{H}^{d-1}(\partial A_r \setminus \partial^+ A_r) = 0$;

(iii) $\mathcal{H}^{d-1}(\partial A_r \setminus \mathrm{Unp}\,(A)) = 0$ and $\mathcal{H}^{d-1}(\partial A_r \cap \mathrm{Unp}\,(A) \setminus \partial^+ A_r) = 0$;

(iv) $\mathcal{H}^{d-1}(\partial A_r \setminus \mathrm{Unp}\,(A)) = 0$;

(v) $\mathcal{H}^{d-1}(\partial A_r \cap \mathrm{crit}(A)) = 0$.

Remark:

- (i)$\Leftrightarrow$(v) implies: If V_A is not differentiable at $r > 0$, then r is a critical value of the distance function d_A. ($\mathrm{crit}(A)$ is the set of critical points.)
- These characterizations of differentiability of V_A can also be deduced from the results in **[Hug, Santilli '22]**, not only for Euclidean parallel sets but also for parallel sets w.r.t. other Minkowski norms.

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(v) $\mathcal{H}^{d-1}(\partial A_r \cap \mathrm{crit}(A)) = 0$.

Ideas on the proof:

- (i) $\Leftrightarrow$ (ii): follows from statements on slide 5 and the fact that $(V_A)'_-(r) > (V_A)'_+(r)$ at non-differentiable r;
- (ii) $\Leftrightarrow$ (iii): follows from the disjoint decomposition

$$\partial A_r \setminus \partial^+ A_r = (\partial A_r \setminus \partial^+ A_r) \setminus \mathrm{Unp}\,(A) \cup (\partial A_r \setminus \partial^+ A_r) \cap \mathrm{Unp}\,(A)$$

  and $\partial^+ A_r \subset \mathrm{Unp}\,(A)$;
- (iii) $\Leftrightarrow$ (iv): follows from $\mathcal{H}^{d-1}(\partial A_r \cap (\mathrm{Unp}\,(A) \setminus \partial^+ A_r)) = 0$ for all r.
- (iv) $\Leftrightarrow$ (v): follows from $\mathcal{H}^{d-1}(\partial A_r \cap (\mathrm{reg}(A) \setminus \mathrm{Unp}\,(A))) = 0$ for all r. □

Relation with critical values

Lemma

For any $A \subset \mathbb{R}^d$ compact and $r > 0$,

$$\partial A_r \cap \partial^+ A_r \subset \partial A_r \cap \mathrm{Unp}\,(A) \subset \partial A_r \cap \mathrm{reg}(A).$$

Further, for any $r > 0$, both inclusions are equalities up to an $\mathcal{H}^{d-1}$-null set, i.e.,

$$\mathcal{H}^{d-1}(\partial A_r \cap (\mathrm{Unp}\,(A) \setminus \partial^+ A_r)) = 0, \tag{1}$$

$$\mathcal{H}^{d-1}(\partial A_r \cap (\mathrm{reg}(A) \setminus \mathrm{Unp}\,(A))) = 0. \tag{2}$$

Idea of proof: The set inclusions are easy (a point outside of A having a unique nearest neighbour in A is regular).
For a regular point $x \in \partial A_r \cap \mathrm{reg}(A)$, the boundary ∂A_r is a Lipschitz surface in some neighbourhood of x (see **[Fu '85]**). Hence, by Rademacher's theorem, the outer normal $n(y)$ is differentiable $\mathcal{H}^{d-1}$-a.e. on this surface, and any point $y \in \partial A_r$ where $n(y)$ is differentiable belongs to $\partial^+ A_r$. These Lipschitz surfaces cover the set $\partial A_r \cap \mathrm{reg}(A)$. □

Characterization of differentiability of V_A

For a compact set $A \subset \mathbb{R}^d$ ($d \in \mathbb{N}$), let

$$N_A := \{r > 0 : V_A \text{ is not differentiable at } r\}$$

be the set of non-differentiability points of the volume function V_A.

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Properties of N_A:

- countable,
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- not necessarily closed.

Question: Which bounded, countable subsets N of $(0, \infty)$ occur as sets N_A of non-differentiability points of some compact set $A \subset \mathbb{R}^d$?

Characterization of non-differentiability sets for $d = 1$

Theorem C [Rataj, W. '23+]

Let $N \subset (0, \infty)$. Then there exists some compact set $A \subset \mathbb{R}$ such that $N_A = N$ if and only if $\sum_{s \in N} s < \infty$.

Note that the condition $\sum_{s \in N} s < \infty$ implies that N is countable and bounded and moreover, that the set N can only accumulate at 0.

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Idea of proof: The set $\mathrm{conv}(A) \setminus A$ consists of at most countably many bounded open intervals. We denote by $\ell_1, \ell_2, \ldots$ the lengths of these intervals in non-increasing order ($(\ell_j)_{j \in \mathbb{N}}$ is called *fractal string* associated with A).

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V_A is not differentiable at some $r > 0$ if and only if $2r$ appears in the fractal string. Hence

$$N_A = \{r \in (0, \infty) : 2r = \ell_j \text{ for some } j \in \mathbb{N}\}.$$

This implies $\sum_{s \in N} s = 2 \sum_{i \in N} \ell_i < \infty$.

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This shows that the condition $\sum_{s \in N} s < \infty$ is necessary. For the sufficiency we construct an example for each such N. □

Characterization of non-differentiability sets for $d = 2$

For a compact set $K \subset \mathbb{R}$ and $\alpha > 0$, the degree-α gap sum of K is

$$G_\alpha(K) := \sum_j \ell_j^\alpha,$$

where (ℓ_j) is the fractal string associated with K (lengths of complementary intervals in non-increasing order).

Theorem D [Rataj, W. '22+]

(i) Let $\varepsilon > 0$ and $N \subset [\varepsilon, \infty)$ be a bounded countable set. Then $N = N_A$ for some compact set $A \subset \mathbb{R}^2$ if and only if $\lambda^1(\overline{N}) = 0$ and $G_{1/2}(\overline{N}) < \infty$.

(ii) Let $N \subset (0, \infty)$ be a bounded countable set. Then $N = N_A$ for some compact set $A \subset \mathbb{R}^2$ if and only if $\lambda^1(\overline{N}) = 0$ and

$$\int_0^\infty G_{1/2}(\overline{N} \cap [r, \infty)) \sqrt{r}\, dr < \infty.$$

The condition in (i) implies $\overline{\mathcal{M}}^{1/2}(\overline{N}) = 0$ (and thus $\overline{\dim}_M \overline{N} \leq \frac{1}{2}$);
the condition in (ii) only implies $\overline{\mathcal{M}}^{4/5}(\overline{N}) = 0$ (and thus $\overline{\dim}_M \overline{N} \leq \frac{4}{5}$).

Some ideas from the proof of Theorem D

Necessity follows from a result by Rataj and Zajíček on critical values of d_A:

Theorem [Rataj, Zajíček '20]

(i) Let $\varepsilon > 0$ and $C \subset [\varepsilon, \infty)$. Then $C \subset \mathrm{cv}(A)$ for some compact set $A \subset \mathbb{R}^2$ if and only if $\lambda^1(\overline{C}) = 0$ and $G_{1/2}(\overline{C}) < \infty$.

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- If $N_A = N$ for some compact set $A \subset \mathbb{R}^2$, then $\overline{N} \subset \mathrm{cv}(A)$. (Recall that $N_A \subset \mathrm{cv}(A)$, and the fact that $\mathrm{cv}(A)$ is closed.)

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- Given $s \in (0, \infty)$, how to construct a set with non-differentiability point at s?
- The *potato-sack problem* (Problem 10.1 from *The Scottish Book*): Given a sequence of compact, convex sets (potatos) in $\mathbb{R}^d$ with uniformly bounded diameter and finite total volume, do they fit in a sack of finite size? (first solution: **[Kosiński '57]**)

Examples of non-differentiability sets for $d = 2$

- For $q \in (0, 1/2)$ let $F_q \subset \mathbb{R}$ be the self-similar set generated by the two mappings $x \mapsto qx$ and $x \mapsto qx + (1 - q)$.

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Examples of non-differentiability sets for $d = 2$

- For $q \in (0, 1/2)$ let $F_q \subset \mathbb{R}$ be the self-similar set generated by the two mappings $x \mapsto qx$ and $x \mapsto qx + (1 - q)$.
- Observe that $\{0, 1\} \subset F_q \subset [0, 1]$ and $\dim_M F_q = \frac{\log 2}{\log(1/q)} < 1$.
- Let E_q be the set of all endpoints of intervals of $[0, 1] \setminus F_q$.
- Then $\overline{E}_q = F_q$, $\lambda_1(F_q) = 0$ and for any $\alpha > 0$,

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This shows: sets of non-differentiability points may have any Minkowski dimension between 0 and $1/2$. In fact, any value up to $4/5$ is also possible (but not sets of the above type).

Non-differentiability sets for $d \geq 3$

Proposition [Rataj, W. '23+]

Let $d \in \mathbb{N}$ and $N \subset (0, \infty)$ be such that $\sum_{s \in N} s^d < \infty$. Then there exists some compact set $A \subset \mathbb{R}^d$ such that $N_A = N$.

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