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Oscillatory Integrals and Fractal Dimension

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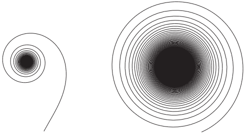
Content

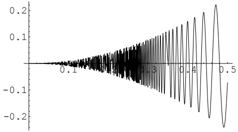
Motivation

- A natural idea is that "density" of an orbit is related to the quantity and quality of objects which could be produced by perturbation of the system.
- Classical fractal analysis associates box dimension and Minkowski content to measurable sets, which in some sense measures the "density" of a set.
- We study continuous systems by
  -spiral trajectories near a focus, limit cycle and polycycle
  -discrete system generated by the Poincaré map
  -discrete system generated by the unit-time map
- We also study oscillatory integrals and singularities of maps, by the curve defined parametrically by the oscillatory integral

Examples from dynamical systems



weak focus

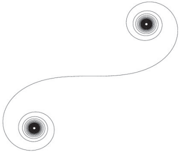


chirp

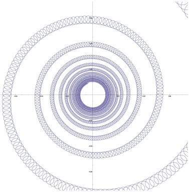
sequence

Examples of curves defined by oscillatory integrals

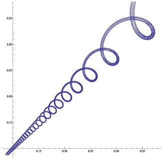
$$I(\tau) = \int_{\mathbb{R}} e^{i\tau f(x)} \Phi(x)\, dx$$



clothoid



phase $f(x) = x^2 + c$



discontinuous $\mathbf{\Phi(x)}$

Definition of upper Minkowski content and upper box dimension

upper s-dimensional Minkowski content of the bounded set $A \in \mathbb{R}^N$, $0 \leq s \leq N$: $\mathcal{M}^{*s}(A) = \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon(A)|}{\varepsilon^{N-s}}$

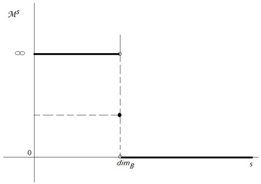


Figure: Minkowski content $\mathcal{M}^{*s}$ as function of $s \in [0, N]$

upper box dimension: $\overline{\dim}_B A = \inf\{s \geq 0 \mid \mathcal{M}^{*s}(A) = 0\}$.

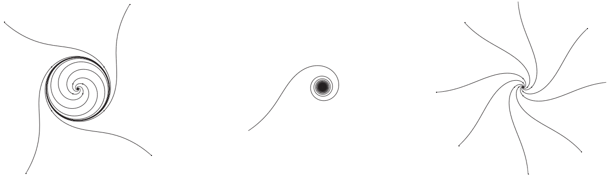
Definition of lower Minkowski content and lower box dimension

- analogously we define lower Minkowski content $\mathcal{M}_*^s$, lower box dimension $\underline{\dim}_B A$
- box dimension $s = \underline{\dim}_B A = \overline{\dim}_B A$
- $F(x)$ and $G(x)$,
  $F(x) \simeq G(x)$, as $x \to 0$, if exist $C_1, C_2, d > 0$ such that
  $C_1 \leq F(x)/G(x) \leq C_2$, $x \in (0, d)$, such functions are comparable
- $\mathcal{M}^{*s}, \mathcal{M}_*^s \neq 0, \infty \Rightarrow |A_\varepsilon(A)| \simeq \varepsilon^{N-s}$, otherwise not comparable to any power of ε

Weak focus in the normal form [Žubrinić, Ž, 2005]

$$\begin{cases} \dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} = 1. \end{cases} \tag{1}$$

Hopf bifurcation occurs for $l = 1$ if $a_0 = 0$. Hopf-Takens bifurcation occurs for $l > 1$, producing l limit cycles in the system



(1) $a_0 < 0$ strong focus $r = 0$ and the limit cycle $r = \sqrt{-a_0}$. Spiral trajectories are exponential, box dimension equal to 1.
(2) $a_0 = 0$ the origin $r = 0$ is weak focus with $\dim_B \Gamma = 4/3$, where $r = (-2\varphi)^{-1/2}$.
(3) $a_0 > 0$ strong focus, exponential spirals, box dimension equal to 1.

Theorem

Γ a part of a trajectory of (1) near the origin.
(a) $a_0 \neq 0$, then the spiral Γ is of exponential type, that is, comparable to $r = e^{a_0 \varphi}$, and hence $\dim_B \Gamma = 1$.
(b) k is fixed, $1 \leq k \leq l$, $a_l = 1$ and $a_0 = \cdots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable to the spiral $r = \varphi^{-1/2k}$, and

$$d := \dim_B \Gamma = \frac{4k}{2k+1}.$$

Singularities of differentiable maps

- We analyze singular points ($\nabla f = 0$) of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$.
- Standard approach is the analysis of asymptotic behavior of oscillatory integrals

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(x)} \Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty,$$

with respect to parameter $\tau \in \mathbb{R}$. Notice $I : \mathbb{R} \to \mathbb{C}$.
- We study geometrical properties of curve in the complex plane generated by $I(\tau)$ for $\tau \geq \tau_0 > 0$, and also of graphs of real and imaginary parts of $I(\tau)$.

$$X(\tau) := \mathrm{Re}\, I(\tau) = \int_{\mathbb{R}^n} \cos(\tau f(\mathbf{x}))\Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty,$$

$$Y(\tau) := \mathrm{Im}\, I(\tau) = \int_{\mathbb{R}^n} \sin(\tau f(\mathbf{x}))\Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty.$$

Standard assumptions on functions f and Φ

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(\mathrm{x})} \Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty, \quad \tau \in \mathbb{R}.$$

- Function $\Phi : \mathbb{R}^n \to \mathbb{R}$
  - is called the *amplitude function*,
  - is of class C^∞,
  - is a non-negative function with compact support,
  - point $\mathbf{0}$ is inside the compact support of function Φ.
- Function $f : \mathbb{R}^n \to \mathbb{R}$
  - is called the *phase function*,
  - point $\mathbf{0} \in \mathbb{R}^n$ is the critical point of function f,
  - is a *real analytic* function in the neighborhood of its critical point $\mathbf{0}$,
  - point $\mathbf{0}$ is *the only* critical point of function f inside the compact support of function Φ.

Oscillatory and curve dimensions

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(\mathrm{x})}\Phi(\mathbf{x})\,d\mathbf{x}, \quad \tau \to \infty,$$

$$X(\tau) := \mathrm{Re}\, I(\tau) = \int_{\mathbb{R}^n} \cos(\tau f(\mathbf{x}))\Phi(\mathbf{x})\,d\mathbf{x}, \quad \tau \to \infty,$$

$$Y(\tau) := \mathrm{Im}\, I(\tau) = \int_{\mathbb{R}^n} \sin(\tau f(\mathbf{x}))\Phi(\mathbf{x})\,d\mathbf{x}, \quad \tau \to \infty.$$

Under the standard assumptions on functions f and Φ we determine:

- *Oscillatory dimensions* of functions $X(\tau)$ and $Y(\tau)$, which are defined as the box dimension of graphs of functions

  $$x(t) := X(1/t), \quad t \to 0, \qquad y(t) := Y(1/t), \quad t \to 0,$$

  and associated Minkowski contents.
- *Curve dimension* of function $I(\tau)$, which is defined as the box dimension of the curve defined in the complex plane by $I(\tau)$, for $\tau \geq \tau_0 > 0$, and associated Minkowski contents.

Phase function of a single variable

Theorem ($n = 1$)

Let the standard assumptions on f and Φ hold, and let $f(0) \neq 0$. Let $f'(0) = f''(0) = \cdots = f^{(p-1)}(0) = 0$ and $f^{(p)}(0) \neq 0$, $p \geq 2$ (p is the order of degeneracy). Then:

- *Oscillatory dimension of both $X(\tau)$ and $Y(\tau)$ is $d' = \frac{3p-1}{2p}$ and associated graphs are Minkowski nondegenerate.*
- *Curve dimension of $I(\tau)$ is $d = \frac{2p}{p+1}$, associated curve Γ is Minkowski measurable, and d-dimensional Minkowski content of Γ is*

$$\mathcal{M}^d(\Gamma) = |C_1|^{\frac{2p}{p+1}} \cdot \pi \cdot \left(\frac{\pi}{p \cdot |f(0)|}\right)^{-\frac{2}{p+1}} \cdot \frac{p+1}{p-1}.$$

Remark. The constant C_1 can be explicitly calculated using the standard formula for phase functions with nondegenerate critical point where $I(\tau) \sim C_1 \cdot e^{i\tau f(0)} \cdot \tau^{-1/p}$, as $\tau \to \infty$.

Newton diagram-$\mathbb{R}$-nondegeneracy

We consider the power series of the phase f

$$f(x) = \sum a_k x^k$$

with real coefficients, having monomials

$$x^k = x_1^{k_1} \dots x_n^{k_n}$$

with multi-index $k = (k_1, \dots k_n)$.

- Polynomial f_Δ that equals to the sum of monomials belonging to the Newton diagram, is called the principal part of the series.
- The principal part f_Δ of the power series f with real coefficients is $\mathbb{R}$-nondegenerate if for every compact face γ of the Newton polyhedron of the series the polynomials

$$\partial f_\gamma / \partial x_1, \dots, \partial f_\gamma / \partial x_n$$

do not have common zeroes in $(\mathbb{R} \setminus 0)^n$.

Newton diagram-remoteness of a critical point

- The bisector intersects the boundary of the Newton polyhedron in exactly one point $(c, \dots, c)$, which is called the center of the Newton polyhedron. The remoteness of the Newton polyhedron is equal to $r = -1/c$. If $r > -1$ the Newton polyhedron is *remote*, which means that it does not contain the point $(1, \dots, 1)$.
- The remoteness of the critical point of the phase is the upper bound of the remotenesses of the Newton polyhedra of the Taylor series of the phase in all systems of local analytic coordinates with origin at the critical point.

Phase function of two variables

Theorem ($n = 2$)

Let $n = 2$, the standard assumptions on f and ϕ hold, and let $f(0) \neq 0$. Let β be the remoteness of the critical point of the phase function f. Let Γ be the curve defined by $X(\tau)$ and $Y(\tau)$, near the origin, with $I(\tau) \sim e^{i\tau f(0)} \left(a_{0,\beta}\tau^{\beta} + a_{1,\beta}\tau^{\beta}\log\tau\right)$ as $\tau \to \infty$. Then:

- *If $a_{1,\beta} = 0$ then the oscillatory dimension of X and Y is equal to*
  $d' = (\beta + 3)/2$
  and Minkowski nondegenerate. The curve dimension of I is
  $d = 2/(1 - \beta)$
  and associated Minkowski content is

$$\mathcal{M}^d(\Gamma) = \left[\frac{|a_{0,\beta}|}{|f(0)|^{\beta}}\right]^{\frac{2}{1-\beta}} \cdot [-\beta]^{\frac{2\beta}{1-\beta}} \cdot \pi^{\frac{1+\beta}{1-\beta}} \cdot \frac{1-\beta}{1+\beta} \qquad (2)$$

- *If $a_{1,\beta} \neq 0$ then oscillatory and curve dimensions are the same as in previous case but Minkowski degenerate.*

Phase of $n > 2$ variables-nondegenerate critical point

Proposition

- *Let $n > 2$ the standard assumptions on f and ϕ hold, and let $f(0) \neq 0$. Let Γ be the curve defined by $X(\tau)$ and $Y(\tau)$, near the origin.*
- *If phase function f has the nondegenerate critical point then oscillatory and curve dimensions are equal to 1.*

Remark. Proof based on the asymptotic expansion with nondegenerate critical point $I(\tau) \sim C \cdot e^{i\tau f(0)} \cdot \tau^{-n/2}$, as $\tau \to \infty$.

Phase of $n > 2$ variables-degenerate critical point

Theorem ($n > 2$)

- *Let phase function f has the degenerate critical point*
  $I(\tau) \sim e^{i\tau f(0)} \sum_{\alpha<\beta} \sum_{k=0}^{n-1} a_{k,\alpha}(\phi)\tau^{\alpha} (\log \tau)^{k}, \quad as\ \tau \to \infty$.
- *Let the Newton diagram of the phase f be $\mathbb{R}$-nondegenerate and remote with remoteness β of the critical point. Then:*
- *If $a_{0,\beta} \neq 0$ and $a_{k,\beta} = 0$, $k = 1, \ldots, n-1$ then the oscillatory dimension of $X(\tau)$ and $Y(\tau)$ is equal to*
  $d' = (\beta + 3)/2$
  and Minkowski nondegenerate. The curve dimension of $I(\tau)$ is
  $d = 2/(1 - \beta)$
  and Minkowski content is given by (2).
- *If for some $L > 0$ holds $a_{L,\beta} \neq 0$, then oscillatory and curve dimensions are the same as in the previous case, and Minkowski degenerate.*

Example homogeneous polynomial

A simple example is a homogeneous polynomial phase of degree $n \geq 2$. The multiplicity $\mu = (n-1)^2$, if the multiplicity is finite. We obtain the curve dimension

$$d = 2 - \frac{4}{n+2},$$

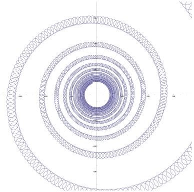
and the relation between the curve dimension and the multiplicity of the singularity of the phase function

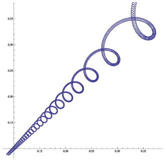
$$d = 2 - \frac{4}{\sqrt{\mu} + 3}.$$

The bigger multiplicity produces the bigger box dimension.

Further research with oscillatory integrals

- Caustics consisting of degenerate singularities are also interesting objects where bifurcations appear
- Case with amplitude which is not C^∞
- Non-analytic phase





Main references

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